Integration

Integration (Primitive or antiderivative)

Integration is the reverse process of differentiation.

If $\frac{d}{dx}[F(x)] = f(x)$, then the integration of $f(x)$ w.r.t x is $\boxed{\int f(x)\,dx = F(x) + C}$

→ The symbol $\int$ is used to denote the operation of integration called as integral sign.

→ The function (here $f(x)$) is called the integrand.

→ 'dx' denote that the integration is to be performed w.r.t. x (x is the variable of integration)

→ 'C' is the constant of integration (which gives family of curves).

→ Integrate means to find the integral of the function and the process is known as integration.

Algebra of integrals

(i) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(ii) $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$, for a constant λ

(iii) $\frac{d}{dx}\left[\lambda \int f(x)\,dx\right] = \lambda \frac{d}{dx}\left(\int f(x)\,dx\right) = \lambda f(x)$

Simple integration formulae

(I) $\int x^n dx = \frac{x^{n+1}}{n+1} + K, \; n \neq -1$

(II) $\int \frac{1}{x} dx = \ln|x| + K$

(III) $\int a^x dx = \frac{a^x}{\ln a} + K$

(IV) $\int e^x dx = e^x + K$

(V) $\int \sin x \, dx = -\cos x + K$

(VI) $\int \cos x \, dx = \sin x + K$

(VII) $\int \sec^2 x \, dx = \tan x + K$

(VIII) $\int \operatorname{cosec}^2 x \, dx = -\cot x + K$

(IX) $\int \sec x \tan x \, dx = \sec x + K$

(X) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + K$

(XI) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + K$

(XII) $-\int \frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + K$

(XIII) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + K$

(XIV) $-\int \frac{1}{1+x^2} dx = \cot^{-1} x + K$

(XV) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + K$

(XVI) $-\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + K$

(XVII) $\int \sinh x \, dx = \cosh x + K$

(XVIII) $\int \cosh x \, dx = \sinh x + K$

(XIX) $\int \operatorname{sech} x^2 dx = \tanh x + K$

~~(XX) $\int \operatorname{cosec} h^2 dx = \tanh x + K$~~

(XX) $\int \operatorname{cosech} x^2 dx = -\coth x + K$

(XXI) $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + K$

(XXII) $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + K$

Methods of Integration

(1) Integration by using standard formulae.

(2) Integration by substitution.

(3) Integration by parts.

(1) Integration by using standard formulae.

Example-1 Evaluate the following

(i) $\int\left(5x^3+2x^5-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

Soln $\int\left(5x^3+2x^5-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

$= 5\int x^3 dx + 2\int x^5 dx - 7\int x\,dx + \int x^{-1/2}dx + 5\int \frac{dx}{x}$

$= 5\times\frac{x^4}{4} + 2\times\frac{x^6}{6} - 7\times\frac{x^2}{2} + \frac{x^{1/2}}{\frac{1}{2}} + 5\ln|x| + K$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7x^2}{2} + 2x^{1/2} + 5\ln x + K$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7x^2}{2} + 2\sqrt{x} + 5\ln x + K$ //.

(ii) $\int\left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx$

Soln $\int\left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx$

$= \int\frac{3x^4}{x^3}dx - \int\frac{5x^3}{x^3}dx + \int\frac{4x^2}{x^3}dx - \int\frac{x}{x^3}dx + \int\frac{2}{x^3}dx$

$= 3\int x\,dx - 5\int dx + 4\int\frac{dx}{x} - \int x^{-2}dx + 2\int x^{-3}dx$

$= 3\times\frac{x^2}{2} - 5x + 4\ln x + \frac{1}{x} - \frac{1}{x^2} + K$

$= \frac{3x^2}{2} - 5x + 4\ln x + \frac{1}{x} - \frac{1}{x^2} + K$,

(iii) $\int \left(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right) dx$

Solⁿ $\int \left(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right) dx$

$= 4\int \cos x\, dx - 3\int e^x dx + 2\int \frac{dx}{\sqrt{1-x^2}}$

$= 4\sin x - 3e^x + 2\sin^{-1}x + K$ //

(iv) $\int 6x^3 (x+5)^2 dx$

Solⁿ $\int 6x^3 (x+5)^2 dx$

$= \int 6x^3 (x^2 + 10x + 25) dx$

$= \int (6x^5 + 60x^4 + 150x^3) dx$

$= 6\int x^5 dx + 60\int x^4 dx + 150\int x^3 dx$

$= 6 \times \frac{x^6}{6} + 60 \times \frac{x^5}{5} + 150 \times \frac{x^4}{4} + K$

$= x^6 + 12x^5 + \frac{75}{2}x^4 + K$ //

(v) $\int 5\tan^2 x\, dx$

Solⁿ $\int 5\tan^2 x\, dx$

$= \int 5(\sec^2 x - 1) dx$

$= 5\int \sec^2 x\, dx - 5\int 1.dx$

$= 5\tan x - 5x + K$ //

(vi) $\int \sin^2 \frac{x}{2} dx$

Solⁿ $\int \sin^2 \frac{x}{2} dx$

$= \int \left(\frac{1-\cos x}{2}\right) dx = \frac{1}{2}\left[\int dx - \int \cos x\, dx\right]$

$= \frac{1}{2}[x - \sin x] + K$ //

(vii) $\int \frac{\sin x}{1+\sin x}\,dx$

Sol^n $\int \frac{\sin x}{1+\sin x}\,dx$

$= \int \frac{(1-\sin x)\sin x}{(1+\sin x)(1-\sin x)}\,dx$

$= \int \frac{(1-\sin x)\sin x}{1-\sin^2 x}\,dx$

$= \int \left(\frac{\sin x - \sin^2 x}{\cos^2 x}\right) dx = \int \frac{\sin x}{\cos^2 x}\,dx - \int \tan^2 x\,dx$

$= \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx - \int (\sec^2 x - 1)\,dx$

$= \int \tan x \cdot \sec x\,dx - \int \sec^2 x\,dx + \int dx$

$= \sec x - \tan x + x + k$ //

(viii) $\int \frac{1}{\sin^2 x \cdot \cos^2 x}\,dx$

Sol^n $\int \frac{1}{\sin^2 x \cos^2 x}\,dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx$

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\,dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\,dx$

$= \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx$

$= \tan x - \cot x + k$ //

(ix) $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx$

Sol^n $\int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\} dx$ $\quad$ $\left(\because 1-\cos 2x = 2\sin^2 x,\ 1+\cos 2x = 2\cos^2 x\right)$

$= \int \tan^{-1}(\tan x)\,dx$ $\quad$ $(\because \tan^{-1}(\tan x) = x)$

$= \int x\,dx = \frac{x^2}{2} + k$

Exercise

Evaluate the following

(i) $\int \frac{1}{x\sqrt{x}} dx$

(ii) $\int \left(x^{\frac{4}{7}} + \frac{1}{x^{1/3}}\right) dx$

(iii) $\int \frac{1-\sin^2 x}{\sin^2 x} dx$

(iv) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$

(v) $\int \sqrt{1+\sin 2x}\, dx$

(vi) $\int \frac{e^{2x}+1}{e^x} dx$

(vii) $\int e^{2\ln x} dx$

(viii) $\int \left(\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\right) dx$

(ix) $\int \frac{x^2 + \sqrt{x^2-1}}{x^3\sqrt{x^2-1}} dx$

(x) $\int \frac{1-\cos 2x}{1+\cos 2x} dx$

(2) Integration by substitution

The integral $\int f(x)\,dx$ can be reduced to another form by changing the independent variable 'x' by 't'

substitute $x = \phi(t)$

$$\int f(x)\,dx = \int f(x)\left(\frac{dx}{dt}\right) \cdot dt = \int f[\phi(t)]\, \phi'(t)\, dt,$$

where $x = \phi(t)$

Type-I $\int f(ax+b)\,dx$

Put $ax+b=t$

$a\,dx = dt$

$\Rightarrow dx = \frac{1}{a}dt$

$\therefore \int f(ax+b)\,dx$

$= \int f(t)\frac{1}{a}dt = \frac{1}{a}\int f(t)\,dt.$

Type-II $\int x^{n-1} f(x^n)\,dx$

Put $x^n = t$

$nx^{n-1}dx = dt$

$\Rightarrow x^{n-1}dx = \frac{dt}{n}$

$\therefore \int x^{n-1} f(x^n)\,dx$

$= \int f(t)\frac{dt}{n} = \frac{1}{n}\int f(t)\,dt.$

Type-III $\int \{f(x)\}^n \cdot f'(x)\,dx$

put $f(x) = t$

Differentiate both sides w.r.t x

$f'(x) = \frac{dt}{dx}$

$\Rightarrow f'(x)\,dx = dt$

$\therefore \int [f(x)]^n \cdot f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1} + c$

$= \frac{[f(x)]^{n+1}}{n+1} + c \quad (\because t = f(x))$

Type-IV $\int \frac{f'(x)}{f(x)}dx$

put $f(x) = t$

$\Rightarrow f'(x)\,dx = dt$

$\therefore \int \frac{f'(x)}{f(x)}dx$

$= \int \frac{dt}{t} = \ln|t| + c = \ln|f(x)| + c \quad (\because f(x) = t).$

Some useful results

① $\int \frac{dx}{ax+b}$

Sol^n put $ax+b = t$

Differentiate both sides w.r.t x

$$a = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{dt}{a}$$

$\therefore \int \frac{dx}{ax+b}$

$$= \int \frac{dt/a}{t} = \frac{1}{a}\int \frac{dt}{t} = \frac{1}{a}\ln|t| = \frac{1}{a}\ln|ax+b| + c$$

①

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c}$$ Formulae.

② $\int \cot x \, dx$

Sol^n $\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} dx$$

Put $\sin x = t$

Differentiate both sides w.r.t x

$$\cos x = \frac{dt}{dx}$$

$$\Rightarrow dt = \cos x \, dx$$

$\therefore \int \frac{\cos x \, dx}{\sin x}$

$$= \int \frac{dt}{t} = \ln|t| = \ln|\sin x| + c$$

$$\boxed{\int \cot x \, dx = \ln|\sin x| + c}$$ F.

③ $\int \tan x \, dx$

Sol $\int \tan x \, dx$

$$= \int \frac{\sec x \tan x}{\sec x} dx \quad \text{(Multiply \& divide by } \sec x\text{)}$$

put $\sec x = t$

Differentiate both sides w.r.t x.

$$\sec x \tan x = \frac{dt}{dx}$$

$$\Rightarrow \sec x \tan x \, dx = dt$$

$$\int \frac{\sec x \tan x}{\sec x} dx$$

$$= \int \frac{dt}{t} = \ln|t| = \ln|\sec x| + c$$

$$\boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

④ $\int \operatorname{cosec} x \, dx$

Sol $\int \operatorname{cosec} x \, dx$

$$= \int \frac{1}{\sin x} dx$$

$$= \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx$$

Divide numerator & denominator by $\cos^2 \frac{x}{2}$.

$$\int \frac{\frac{1}{\cos^2 x/2}}{\frac{2 \sin \frac{x}{2} \cos x/2}{\cos^2 x/2}} dx$$

$$= \int \frac{\sec^2 x/2}{2 \tan x/2} dx$$

let $\tan \frac{x}{2} = t$

$$\Rightarrow \sec^2 x/2 \times \frac{1}{2} dx = dt$$

$$\Rightarrow \sec^2 x/2 \, dx = 2 dt$$

$= \int \frac{\sec^2 x/2}{2\tan\frac{x}{2}} dx$ becomes

$= \int \frac{2\,dt}{2t} = \int \frac{dt}{t} = \ln|t| + c$

$= \ln|\tan\frac{x}{2}| + c$ //

$\boxed{\int \operatorname{cosec} x\, dx = \ln|\tan\frac{x}{2}| + c}$ F

(5) $\int \sec x\, dx$

Sol $\int \sec x\, dx$

$= \int \operatorname{cosec}(\frac{\pi}{2}+x)\, dx$ $(\because \operatorname{cosec}(\pi/2+x) = \sec x)$

$= \ln|\tan(\frac{\pi}{4}+\frac{x}{2})| + c$ // $(\because \int \operatorname{cosec} x\, dx = \ln|\tan\frac{x}{2}| + c)$

$\boxed{\int \sec x\, dx = \ln|\tan(\frac{\pi}{4}+\frac{x}{2})| + c}$ F

By applying above formula. we obtain following

(1) $\boxed{\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c.}$ F

Pf $\int \cos(ax+b)\, dx$

Put $ax+b = \theta$

Differentiate both sides w.r.t x.

$a = \frac{d\theta}{dx}$

$\Rightarrow a\,dx = d\theta \Rightarrow dx = \frac{d\theta}{a}$

$\therefore \int \cos(ax+b)\, dx$

$= \int \cos\theta \times \frac{d\theta}{a}$

$= \frac{1}{a}\int \cos\theta\, d\theta = \frac{1}{a}\sin\theta + c$

$= \frac{1}{a}\sin(ax+b) + c$ //

Similarly prove the following

② $\boxed{\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c}$ LF

③ $\boxed{\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c}$ LF

④ $\boxed{\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+c}$ LF

⑤ $\boxed{\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c}$ LF

⑥ $\boxed{\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c}$ LF

⑦ $\boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c}$ LF

⑧ $\boxed{\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{1}{a}\sin^{-1}(ax+b)+c}$ LF

⑨ $\boxed{\int \frac{dx}{1+(ax+b)^2} = \frac{1}{a}\tan^{-1}(ax+b)+c}$ LF

⑩ $\boxed{\int (ax+b)^n\,dx,\ n\neq -1 = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c}$ LF

Ex-1 Integrate the following

① $\int x\sin x^2\,dx$

Soln Given $\int x\sin x^2\,dx$

Let $x^2 = z$

~~Differentiate~~ both side w.r.t z.

~~2x dx = dz~~ $2x = \frac{dz}{dx}$

$\Rightarrow 2x\,dx = dz \rightarrow x\,dx = \frac{dz}{2}$

$\Rightarrow \int x \sin x^2 \, dx$

$\Rightarrow \int \sin z \cdot \frac{dz}{2}$

$\Rightarrow \frac{1}{2} \int \sin z \, dz$

$\Rightarrow -\frac{1}{2} \cos z + c$

$\Rightarrow -\frac{1}{2} \cos x^2 + c$

(2) $\int (x-2)\sqrt{(x^2-4x+7)}\, dx$

Soln $\int (x-2)\sqrt{(x^2-4x+7)}\, dx$

Let $x^2-4x+7 = t^2$ $\Rightarrow \cancel{t = \sqrt{x^2-4x+7}}$

$\Rightarrow$ Differentiate both sides w.r.t x

$2x-4 = 2t\frac{dt}{dx}$

$\Rightarrow (2x-4)\,dx = 2t\,dt$

$\Rightarrow \cancel{2}(x-2)\,dx = \cancel{2}t\,dt$

$\Rightarrow (x-2)\,dx = t\,dt$

$\int (x-2)\sqrt{x^2-4x+7}\, dx$

$= \int \sqrt{t^2}\, t\, dt$

$= \int t \times t\, dt \Rightarrow \int t^2\, dt = \frac{t^3}{3} + c$

$= \frac{(x^2-4x+7)^{3/2}}{3} + c$ $\left(\because t = \sqrt{x^2-4x+7} = (x^2-4x+7)^{\frac{1}{2}}\right)$

(3) $\int (3x+5)^7\, dx$

Soln $\int (3x+5)^7\, dx$

put $3x+5 = t$

Diff w.r.t x

$3 = \frac{dt}{dx} \Rightarrow 3\,dx = dt \Rightarrow dx = \frac{1}{3}\,dt$

$\therefore \int (3x+5)^7 dx$

$= \frac{1}{3}\int t^7 dt$

$= \frac{1}{3} \times \frac{t^8}{8} + c$

$= \frac{1}{3} \times \frac{(3x+5)^8}{8} + c = \frac{(3x+5)^8}{24} + c$ //

(4) $\int \frac{x^4+4x^3}{x^5+5x^4+7} dx$

Sol^n $\int \frac{x^4+4x^3}{x^5+5x^4+7} dx$

Put $x^5+5x^4+7 = \theta$

Diff. w.r.t x

$5x^4 + 20x^3 = \frac{d\theta}{dx}$

$\Rightarrow (5x^4+20x^3)dx = d\theta$

$\Rightarrow 5(x^4+4x^3)dx = d\theta$

$\Rightarrow (x^4+4x^3)dx = \frac{d\theta}{5}$

$\therefore \int \frac{x^4+4x^3}{x^5+5x^4+7} dx = \frac{1}{5}\int \frac{d\theta}{\theta} = \frac{1}{5}\ln|\theta| + c$

$= \frac{1}{5}\ln|x^5+5x^4+7| + c$ //

(5) $\int \sin^7 x \cos x \, dx$

Sol^n $\int \sin^7 x \cos x \, dx$

Put $\sin x = \theta$

Diff. both sides w.r.t x

$\cos x = \frac{d\theta}{dx}$

$\Rightarrow \cos x \, dx = d\theta$

$\therefore \int \sin^7 x \cos x \, dx = \int \theta^7 d\theta = \frac{\theta^8}{8} + c$

$= \frac{\sin^8 x}{8} + c$ //

(6) $\int 2e^{\tan^2 x} \tan x \sec^2 x \, dx$

Sol^n $\int 2e^{\tan^2 x} \tan x \sec^2 x \, dx$

put $\tan^2 x = \theta$

Diff. both sides w.r.t x

$2\tan x . \sec^2 x = \frac{d\theta}{dx}$

$\Rightarrow 2\tan x \sec^2 x \, dx = d\theta$

$\therefore \int 2e^{\tan^2 x} \sec^2 x \, dx = \int e^{\theta} d\theta$

$= e^{\theta} + c$

$= e^{\tan^2 x} + c$ //

(7) $\int \frac{3(\ln x)^2}{x} dx$

Sol^n ~~Red~~ $\int \frac{3(\ln x)^2}{x} dx$

put $\ln x = z$

Diff. w.r.t x

$\frac{1}{x} = \frac{dz}{dx} \Rightarrow \frac{dx}{x} = dz$

$\therefore \int \frac{3(\ln x)^2}{x} dx = 3\int z^2 dz = \cancel{3} \times \frac{z^3}{\cancel{3}} + c = (\ln x)^3 + c$ //

Exercises Evaluate the following

(1) $\int \frac{x^2\,dx}{(1+x^2)^2}$

(2) $\int \sec^2(3x+5)\,dx$

(3) $\int \frac{(\tan^{-1}x)^3}{1+x^2}\,dx$

(4) $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$

(5) $\int \tan^3 x \sec^2 x\,dx$

Hints take $\tan x = v$.

(6) $\int \sqrt{1-\sin x}\,dx$

(7) $\int x\sqrt{x^2+3}\,dx$

Hints take $x^2+3=v$.

(8) $\int \frac{7\,dx}{2-3x}$

(9) $\int \frac{x\,dx}{\sqrt{x^2-a^2}}$

(10) $\int \frac{e^x}{(e^x-2)^2}\,dx$

(11) $\int e^{x^3} x^2\,dx$

(12) $\int e^{\cos^2 x} \sin 2x\,dx$

(13) $\int 2x \cot(x^2+3)\,dx$

(14) $\int e^x \tan e^x\,dx$

(15) Imp $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

(16) Imp $\int 3^x e^{2x}\,dx$

(17) Imp $\int \frac{\sin x}{\sin(x+\alpha)}\,dx$

Hints take $x+\alpha=z$.

(18) Imp $\int \frac{\tan x+\tan\alpha}{\tan x-\tan\alpha}\,dx$

Hints take $x-\alpha=z$.

Integration of some Trigonometric functions

If the integrand is of the form $\sin mx \cos nx$, $\sin mx \sin nx$ or $\cos mx \cos nx$, a trigonometric transformation will help to reduce it to the sum of sines or cosines of multiple angles which can be easily integrated.

$$\sin mx \cos nx = \frac{1}{2} \times 2 \sin mx \cos nx$$

$$= \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x \right]$$

Ex-1 Evaluate $\int \sin 3x \cos 2x \, dx$

Sol^n $\int \sin 3x \cos 2x \, dx$

$= \frac{1}{2} \int 2 \sin 3x \cos 2x \, dx$

$= \frac{1}{2} \int \sin(3x+2x) + \sin(3x-2x) \, dx$

$= \frac{1}{2} \int (\sin 5x + \sin x) \, dx$ $\quad$ ($\because 2 \sin x \cos y = \sin(x+y) + \sin(x-y)$)

$= \frac{1}{2} \int \sin 5x \, dx + \frac{1}{2} \int \sin x \, dx$

$= \frac{1}{2} \times \frac{-\cos 5x}{5} + \frac{1}{2} \times -\cos x + c.$

$= \frac{-1}{10} \cos 5x - \frac{1}{2} \cos x + c$

$= \frac{-1}{10} (\cos 5x + 5 \cos x) + c$ //

(ii) Evaluate $\int \sin 2x \sin x \, dx$

Sol^n $\int \sin 2x \sin x \, dx$

$= \frac{1}{2} \int 2 \sin 2x \sin x \, dx$

$= \frac{1}{2} \int \cos(2x-x) - \cos(2x+x) \, dx$

($\because 2 \sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$)

$= \frac{1}{2} \int (\cos x - \cos 3x) \, dx$

$= \frac{1}{2} \int \cos x \, dx - \frac{1}{2} \int \cos 3x \, dx$

$= \frac{1}{2} \times \sin x - \frac{1}{2} \times \frac{\sin 3x}{3} + c$

$= \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$

$= \frac{1}{6}(3\sin x - \sin 3x) + c$

(iii) Evaluate $\int \cos 4x \cos 3x\, dx$

Solⁿ $\int \cos 4x \cos 3x\, dx$

$= \frac{1}{2}\int 2\cos 4x \cos 3x\, dx$

$= \frac{1}{2}\int \cos(4x-3x) + \cos(4x+3x)\, dx$

$(\because 2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta))$

$= \frac{1}{2}\int (\cos x + \cos 7x)\, dx$

$= \frac{1}{2}\int \cos x\, dx + \frac{1}{2}\int \cos 7x\, dx$

$= \frac{1}{2}\sin x + \frac{1}{2} \times \frac{\sin 7x}{7} + c$

$= \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + c$

$= \frac{1}{14}(\sin 7x + 7\sin x) + c$

(iv) Evaluate $\int \sin^2 x\, dx$

Solⁿ $\int \sin^2 x\, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right) dx \quad (\because \sin^2 x = \frac{1-\cos 2x}{2})$

$= \frac{1}{2}\int (1-\cos 2x)\, dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2} \times x - \frac{1}{2} \times \frac{\sin 2x}{2} + c$

$= \frac{x}{2} - \frac{\sin 2x}{4} + c = \frac{1}{4}(2x - \sin 2x) + c$

(v) Evaluate $\int \cos^3 x\,dx$

Sol $\int \cos^3 x\,dx$

$= \int \left(\frac{\cos 3x + 3\cos x}{4}\right) dx$

$= \frac{1}{4}\int (\cos 3x + 3\cos x)\,dx$ $\left(\because \cos 3x = 4\cos^3 x - 3\cos x \Rightarrow 4\cos^3 x = \cos 3x + 3\cos x \Rightarrow \cos^3 x = \frac{\cos 3x + 3\cos x}{4}\right)$

$= \frac{1}{4}\int \cos 3x\,dx + \frac{1}{4}\int 3\cos x\,dx$

$\cancel{= \frac{1}{4} \times \frac{\sin 3x}{3} + \frac{3}{4} \times \int \cos x\,dx}$

$= \frac{1}{4}\int \cos 3x\,dx + \frac{3}{4}\int \cos x\,dx$

$= \frac{1}{4} \times \frac{\sin 3x}{3} + \frac{3}{4} \times \sin x + c$

$= \frac{\sin 3x}{12} + \frac{3\sin x}{4} + c$

$= \frac{1}{12}(\sin 3x + 9\sin x) + c$ //

(vi) Evaluate $\int \cos^5 x\,dx$

Sol $\int \cos^5 x\,dx$

$= \int \cos^4 x \cdot \cos x\,dx$

$= \int (\cos^2 x)^2 \cos x\,dx$

$= \int (1-\sin^2 x)^2 \cos x\,dx$

put $\sin x = \theta$

Diff. w.r.t x

$\cos x = \frac{d\theta}{dx}$

$\Rightarrow d\theta = \cos x\,dx$

$\therefore \int (1-\sin^2 x)^2 \cos x\,dx$

$= \int (1-\theta^2)^2 d\theta = \int (1 - 2\theta^2 + \theta^4)\,d\theta$

$= \int d\theta - 2\int \theta^2 d\theta + \int \theta^4 d\theta$

$= \theta - 2\times\frac{\theta^3}{3} + \frac{\theta^5}{5} + c$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$ //

(vii) Evaluate $\int \sin^4 x \cos^3 x \, dx$

Sol^n $\int \sin^4 x \cos^3 x \, dx$

$= \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x (1-\sin^2 x) \cos x \, dx$

put $\sin x = \theta$

Differentiate both sides w.r.t x

$\cos x = \frac{d\theta}{dx}$

$\Rightarrow d\theta = \cos x \, dx$

$\therefore \int \sin^4 x (1-\sin^2 x)\cos x \, dx$

$= \int \theta^4 (1-\theta^2) d\theta$

$= \int (\theta^4 - \theta^6) \, d\theta = \int \theta^4 d\theta - \int \theta^6 d\theta$

$= \frac{\theta^5}{5} - \frac{\theta^7}{7} + c$

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$ //

(viii) Evaluate $\int \frac{\cos^3 x}{\sin^4 x} dx$

Sol^n $\int \frac{\cos^3 x}{\sin^4 x} dx$

$= \int \frac{\cos^2 x \cdot \cos x}{\sin^4 x} dx$

$= \int \frac{(1-\sin^2 x)}{\sin^4 x} \cos x \, dx$

put $\sin x = \theta$

Diff both sides w.r.t x

$\cos x \, dx = d\theta$

$\therefore \int \frac{(1-\sin^2 x)}{\sin^4 x} \cos x \, dx = \int \frac{1-\theta^2}{\theta^4} d\theta = \int (\theta^{-4} - \theta^{-2}) d\theta$

$= \frac{\theta^{-3}}{-3} - \frac{\theta^{-1}}{-1} + c$

$= -\frac{1}{3\theta^3} + \frac{1}{\theta} + c = \frac{1}{\sin x} - \frac{1}{3\sin^3 x} + c$

$= \operatorname{cosec} x - \frac{1}{3}\operatorname{cosec}^3 x + C$ //

Exercise Evaluate the following:

1. $\int \sin 4x \cos 3x \, dx$
2. $\int \cos 5x \cos 2x \, dx$
3. $\int \sin 6x \sin 3x \, dx$

Imp 4. $\int \sin \frac{3x}{4} \cos \frac{x}{2} \, dx$

5. $\int \cos 2x \cos \frac{x}{2} \, dx$
6. $\int \sin^5 x \, dx$
7. $\int \cos^7 x \, dx$
8. $\int \sin^6 x \, dx$
9. $\int \cos^5 x \sin^3 x \, dx$

Imp 10. $\int \frac{\sin^3 x}{\cos^6 x} \, dx$

11. $\int \sin^4 x \cos^4 x \, dx$
12. $\int \tan^5 \theta \cdot \sec^4 \theta \, d\theta$
13. $\int \tan^5 \theta \, d\theta$

Imp 14. $\int \frac{\sin 4x - \sin 2x}{\cos 5x} \, dx$

Integration by trigonometric substitution

Trigonometric identities

$1-\sin^2\theta = \cos^2\theta$ (or $1-\cos^2\theta = \sin^2\theta$)

$\tan^2\theta + 1 = \sec^2\theta$ (also $\cot^2\theta + 1 = \csc^2\theta$)

$\sec^2\theta - 1 = \tan^2\theta$ (also $\csc^2\theta - 1 = \cot^2\theta$)

→ The integrand of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$ can be simplified by putting $\boxed{x = a\sin\theta}$

$\boxed{x = a\tan\theta}$

$\boxed{x = a\sec\theta}$

$\boxed{x = a\cos\theta}$

$\boxed{x = a\cot\theta}$

$\boxed{x = a\csc\theta}$

→ The integrand of the form x^2+a^2, can be simplify by putting $x = a\tan\theta$ (or $x = a\cot\theta$).

Ex-1 Integrate $\int \frac{dx}{\sqrt{a^2-x^2}}$

Sol: $\int \frac{dx}{\sqrt{a^2-x^2}}$

Let $x = a\sin\theta$

Differentiate both sides w.r.t x

$dx = a\cos\theta\, d\theta$

$x = a\sin\theta$

$\Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta}\, d\theta = \int d\theta$

$= \theta + c$

$= \sin^{-1}\frac{x}{a} + c$ //

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c}$$ IF

Ex-2 Integrate $\int \frac{dx}{x^2+a^2}$

Sol: $\int \frac{dx}{x^2+a^2}$

Let $x = a\tan\theta$

Diff. both sides w.r.t x

$dx = a\sec^2\theta\, d\theta$

$x = a\tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\frac{x}{a}}$

$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{\cancel{dx}\ a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$

$= \int \frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta+1)}$

$= \int \frac{\sec^2\theta\, d\theta}{a\sec^2\theta}$

$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$ F

Ex-3 Integrate $\int \frac{dx}{\sqrt{x^2+a^2}}$

Sol: $\int \frac{dx}{\sqrt{x^2+a^2}}$

Let $x = a\tan\theta$

Diff w.r.t x

$dx = a\sec^2\theta\, d\theta$

~~$x = a\tan\theta$~~

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2(\tan^2\theta+1)}}$

$= \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta}$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + c \quad (\because \int \sec x \, dx = \ln|\sec x + \tan x| + k)$

$= \ln\left|\sqrt{\frac{x^2}{a^2}+1} + \frac{x}{a}\right| + c$

$x = a\tan\theta$

$\Rightarrow \tan\theta = \frac{x}{a}$

We know that

$\sec^2\theta = \tan^2\theta + 1$

$= \frac{x^2}{a^2} + 1$

$\Rightarrow \sec\theta = \sqrt{\frac{x^2}{a^2}+1}$

$= \ln\left|\sqrt{\frac{x^2+a^2}{a^2}} + \frac{x}{a}\right| + c$

$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + c$

$= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + c$

$= \ln\left|x+\sqrt{x^2+a^2}\right| - \ln|a| + c$

$= \ln\left|x+\sqrt{x^2+a^2}\right| + K$ // $(\because$ where $K = c - \ln|a|)$

$\therefore \boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + K}$

Ex-4 Integrate $\int \frac{dx}{\sqrt{x^2-a^2}}$

Soln $\int \frac{dx}{\sqrt{x^2-a^2}}$

Let $x = a\sec\theta$

Diff. w.r.t x

$dx = a\sec\theta\tan\theta \, d\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta \, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{a\sec\theta\tan\theta \, d\theta}{\sqrt{a^2(\sec^2\theta - 1)}}$

$= \int \frac{a\sec\theta\tan\theta \, d\theta}{a\tan\theta}$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + c$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c$

$x = a\sec\theta$
$\Rightarrow \sec\theta = x/a$
$\tan\theta = \sqrt{\frac{x^2}{a^2} - 1}$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2 - a^2}{a^2}}\right| + c$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + c$

$= \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c$

$= \ln|x + \sqrt{x^2 - a^2}| - \ln|a| + c$

$= \ln|x + \sqrt{x^2 - a^2}| + k \quad (\because k = c - \ln|a|)$

$$\therefore \boxed{\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + k}$$

Ex-5 Integrate $\int \frac{dx}{x\sqrt{x^2 - a^2}}$

Sol^n $\int \frac{dx}{x\sqrt{x^2 - a^2}}$

Let $x = a\sec\theta$

Diff both sides w.r.t x

$dx = a\sec\theta\tan\theta \, d\theta$

$x = a\sec\theta$

$\Rightarrow \boxed{\theta = \sec^{-1}\frac{x}{a}}$

$\therefore \int \frac{dx}{x\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta \, d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{a\sec\theta\tan\theta \, d\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta - 1)}}$

$$= \int \frac{a\sec\theta\tan\theta}{a\sec\theta . a\tan\theta} d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{a^2\sec\theta\tan\theta} d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

$$\therefore \boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$$

F

Q.6 Integrate $\int \frac{dx}{x^2-a^2}$

Sol $\int \frac{dx}{x^2-a^2}$

Let $x = a\sec\theta$

Diff. both sides w.r.t x

$dx = a\sec\theta\tan\theta\, d\theta$.

~~$\because \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{\ }}$~~

$$\because \int \frac{dx}{x^2-a^2} = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2(\sec^2\theta-1)}$$

$$= \int \frac{\sec\theta\tan\theta\, d\theta}{a\tan^2\theta}$$

$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \frac{1}{a}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\, d\theta$$

$$= \frac{1}{a}\int \frac{1}{\sin\theta}\, d\theta$$

$$= \frac{1}{a}\int \operatorname{cosec}\theta\, d\theta$$

$$= \frac{1}{a}\ln|\operatorname{cosec}\theta - \cot\theta| + c$$

$$= \frac{1}{a}\ln\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{x+a}\sqrt{x-a}}\right| + c$$

$$= \frac{1}{a}\ln\left|\frac{\sqrt{x-a}}{\sqrt{x+a}}\right| + c$$

$x = a\sec\theta$

$\sec\theta = \frac{x}{a}$

$\cos\theta = \frac{a}{x}$

$\sin\theta = \sqrt{1-\cos^2\theta}$

$= \sqrt{1-\frac{a^2}{x^2}}$

$= \frac{\sqrt{x^2-a^2}}{x}$

$\operatorname{cosec}\theta = \frac{x}{\sqrt{x^2-a^2}}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$= \frac{a}{\sqrt{x^2-a^2}}$

$$= \frac{1}{a} \ln \left| \frac{x-a}{x+a} \right|^{\frac{1}{2}} + c$$

$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c \qquad (\because \log m^n = n \log m)$$

$$\therefore \boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c}$$

F

Q.7. Integrate $\int \frac{dx}{a^2-x^2}$

Sol $\int \frac{dx}{a^2-x^2}$

Let $x = a\sin\theta$

Differentiate both sides w.r.t x

$dx = a\cos\theta \, d\theta$

$$\therefore \int \frac{dx}{a^2-x^2} = \int \frac{a\cos\theta \, d\theta}{a^2 - a^2\sin^2\theta}$$

$$= \int \frac{a\cos\theta \, d\theta}{a^2(1-\sin^2\theta)}$$

$$= \int \frac{\cos\theta \, d\theta}{a\cos^2\theta}$$

$$= \frac{1}{a} \int \frac{1}{\cos\theta} d\theta$$

$$= \frac{1}{a} \int \sec\theta \, d\theta$$

$$= \frac{1}{a} \ln |\sec\theta + \tan\theta| + c$$

$$= \frac{1}{a} \ln \left| \frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}} \right| + c$$

$$= \frac{1}{a} \ln \left| \frac{a+x}{\sqrt{a^2-x^2}} \right| + c$$

$$= \frac{1}{a} \ln \left| \frac{a+x}{\sqrt{a+x}\sqrt{a-x}} \right| + c$$

$x = a\sin\theta$

$\sin\theta = \frac{x}{a}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-\frac{x^2}{a^2}}$

$= \frac{\sqrt{a^2-x^2}}{a}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{a}{\sqrt{a^2-x^2}}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{a^2-x^2}}$

$$= \frac{1}{a} \ln \left| \frac{\sqrt{a+x}}{\sqrt{a-x}} \right| + c$$

$$= \frac{1}{a} \ln \left| \frac{a+x}{a-x} \right|^{1/2} + c \qquad (\because \log m^n = n \log m)$$

$$= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$$

$$\therefore \boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c}$$

Ex-8 integrate $\int \frac{dx}{\sqrt{25-16x^2}}$

Sol^n $\int \frac{dx}{\sqrt{25-16x^2}}$

$= \frac{1}{4}\int \frac{dx}{\sqrt{\left(\frac{5}{4}\right)^2 - x^2}}$

$\int \frac{dx}{\sqrt{25-16x^2}}$

$= \int \frac{dx}{\sqrt{16\left(\frac{25}{16}-x^2\right)}}$

$= \frac{1}{4}\int \frac{dx}{\sqrt{\left(\frac{5}{4}\right)^2 - x^2}}$

$= \frac{1}{4} \sin^{-1} \frac{x}{5/4} + c$ (using formula $\int \frac{dx}{\sqrt{a^2-x^2}}$)

$= \frac{1}{4} \sin^{-1} \frac{4x}{5} + c$ //

Ex-9 $\int \frac{e^x}{e^{2x}+9} dx$

Sol^n $\int \frac{e^x}{e^{2x}+9} dx$

$= \int \frac{e^x}{(e^x)^2+3^2} dx$

let $e^x = z$

$e^x dx = dz$.

$\therefore \int \frac{e^x}{(e^x)^2+3^2} dx = \int \frac{dz}{z^2+3^2} = \frac{1}{3}\tan^{-1}\frac{z}{3} + c$

$= \frac{1}{3}\tan^{-1}\left(\frac{e^x}{3}\right) + c$ //

Ex-40 Integrate $\int \frac{dx}{x\sqrt{x^8-4}}$

Soln $\int \frac{dx}{x\sqrt{x^8-4}}$

$= \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}dx$

$= \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}dx$

Let $x^4 = z$

Diff. w.r.t x

$4x^3 = \frac{dz}{dx}$

$\Rightarrow dz = 4x^3 dx$

$\therefore \frac{1}{4}\int \frac{4x^3\,dx}{x^4\sqrt{x^8-4}}$

$= \frac{1}{4}\int \frac{dz}{z\sqrt{z^2-2^2}} = \frac{1}{4}\times\frac{1}{2}\sec^{-1}\frac{z}{2}+c$ (using formula $\int \frac{dx}{x\sqrt{x^2-a^2}}$)

$= \frac{1}{8}\sec^{-1}\left(\frac{x^4}{2}\right)+c$ //

Ex-41 $\int \frac{x+5}{\sqrt{x^2+6x-7}}dx$.

Soln $\int \frac{x+5}{\sqrt{x^2+6x-7}}dx$

$= \int \frac{x+3+2}{\sqrt{x^2+6x+9-16}}$

$= \int \frac{x+3+2}{\sqrt{(x+3)^2-16}}$

Let $x+3 = z$.

$= \int \frac{z+2}{\sqrt{z^2-16}}dz$

$= \int \frac{z\,dz}{\sqrt{z^2-16}} + 2\int \frac{dz}{\sqrt{z^2-16}}$

$= \sqrt{z^2-16} + 2\int \frac{dz}{z^2-16}$

$= \sqrt{z^2-16} + 2\int \frac{dz}{z^2-4^2}$

$= \sqrt{z^2-16} + 2\ln|z+\sqrt{z^2-16}|+c$

$= \sqrt{z^2-16} + 2\ln|z+\sqrt{z^2-16}|+C$

$\int \frac{z\,dz}{\sqrt{z^2-16}}$

Put $z^2-16=t^2$

$2z\,dz = 2t\,dt$

$\int \frac{t\,dt}{\sqrt{t^2}} = \int \frac{t\,dt}{t}$

$= t = \sqrt{z^2-16}$

$= \sqrt{(x+3)^2-16} + 2\ln\left|x+3+\sqrt{(x+3)^2-16}\right|+C$

$= \sqrt{x^2+6x-7} + 2\ln\left|x+3+\sqrt{x^2+6x-7}\right|+C$ //

Exercises

Integrate the following:

(1) $\int \frac{dx}{11-4x^2}$

(2) $\int \frac{e^{3x}}{\sqrt{4-e^{6x}}}dx$

(3) $\int \frac{dx}{x\sqrt{25-(\ln x)^2}}$

(4) $\int \frac{\cos\theta}{\sqrt{4-\sin^2\theta}}d\theta$

(5) $\int \frac{\cos\theta\, d\theta}{\sqrt{4\sin^2\theta+1}}$

(6) $\int \frac{dx}{\sqrt{5-x^2-4x}}$

(7) $\int \frac{x+3}{\sqrt{5-x^2-4x}}dx$

(8) $\int \frac{dx}{3x^2+7}$

(9) Imp $\int \frac{e^{4x}}{e^{8x}+4}dx$

(10) $\int \frac{\sec\theta\tan\theta}{\sec^2\theta+4}d\theta$

(11) $\int \frac{x^9}{x^{20}+4}dx$

Hint $x^{10}=z$

(12) $\int \frac{dx}{x^2+6x+13}$

(13) $\int \frac{dx}{\sqrt{e^{4x}-5}}$

(14) Imp $\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-4}}$

(15) $\int \frac{dx}{\sqrt{4x^2-6}}$

(16) Imp $\int \frac{dx}{\sqrt{x^2+8x}}$

Hint take $x+4=z$

(17) Imp $\int \frac{x+7}{\sqrt{x^2+8x}}dx$

Integration by parts

If v & w are differentiable function of x, then

$$\frac{d}{dx}(vw) = v\frac{dw}{dx} + w\frac{dv}{dx}$$

or $$v\frac{dw}{dx} = \frac{d}{dx}(vw) - w\frac{dv}{dx}$$

integrating both sides,

$$\int v\frac{dw}{dx}\,dx = \int \frac{d}{dx}(vw)\,dx - \int w\frac{dv}{dx}\,dx$$

$$= wv - \int w\frac{dv}{dx}\,dx$$

Let $u = \frac{dw}{dx}$ & $w = \int u\,dx$, the result can be written as $\int uv\,dx = \left(\int u\,dx\right) \times v - \int\left(\left(\int u\,dx\right) \times \frac{dv}{dx}\right)dx$

this rule is called <u>integration by parts</u>. and is used to integrate the product of two functions

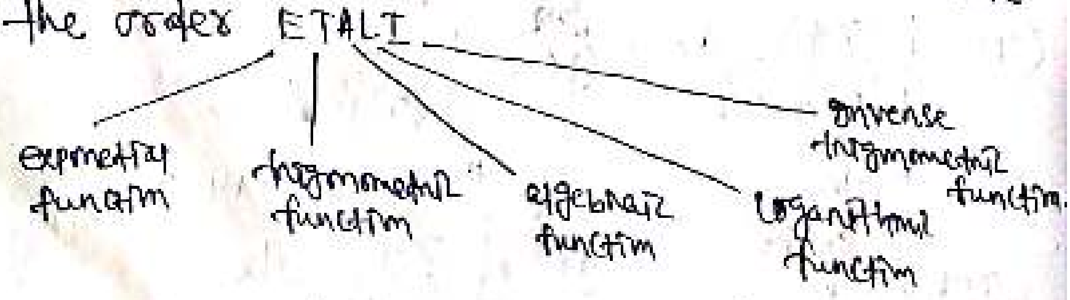
integration of the product of two functions

= (integration of first function) × second function

− integral of (integral of first × derivative of second)

$$\boxed{\text{int. of product} = (\text{int. first}) \times \text{second} - \int (\text{int. first})(\text{der. second})\,dx.}$$

→ The choice of 'first' and 'second' function is made on the order ETALI



function to be integrated	first function	second function
$x^n e^x$	e^x	x^n
$x^n \cos x$	$\cos x$	x^n
$x^n \sin x$	$\sin x$	x^n
$x^n (\ln x)^m$	x^n	$(\ln x)^m$
$x^n \sin^{-1}x$	x^n	$\sin^{-1}x$
$x^n \cos^{-1}x$	x^n	$\cos^{-1}x$
$x^n \tan^{-1}x$	x^n	$\tan^{-1}x$

Example-1

Integrate $\int x \cos x \, dx$

Soln

$$\int \underset{\mathrm{II}}{x}\ \underset{\mathrm{I}}{\cos x}\, dx$$

$$= \left(\int \cos x\, dx\right) \cdot x - \int \left(\int \cos x\, dx\right) \times \frac{dx}{dx} \cdot dx$$

$$= x \sin x - \int \sin x \cdot 1 \cdot dx$$

$$= x \sin x + \cos x + C \,//$$

Example-2 Integrate $\int x^2 e^x \, dx$

Soln $\int \underset{\mathrm{II}}{x^2}\ \underset{\mathrm{I}}{e^x}\, dx$

$$= \left(\int e^x dx\right) \cdot x^2 - \int \left(\int e^x dx\right) \frac{d}{dx}(x^2)\, dx$$

$$= x^2 e^x - \int e^x \times 2x\, dx$$

$$= x^2 e^x - 2\int \underset{\mathrm{II}}{x}\ \underset{\mathrm{I}}{e^x}\, dx$$

$$= x^2 e^x - 2\left[\left(\int e^x dx\right) \cdot x - \int \left(\int e^x dx\right) \cdot 1 \cdot dx\right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= (x^2 - 2x + 2)e^x + C \,//$$

Example-3 Integrate $\int \tan^{-1}x\,dx$

Soln $\int \tan^{-1}x\,dx$

$= \int \underset{I}{1} \times \underset{II}{\tan^{-1}x}\,dx$

$= \left(\int dx\right)\cdot \tan^{-1}x - \int\left(\int dx\right)\cdot \frac{d}{dx}(\tan^{-1}x)\,dx.$

$= x\tan^{-1}x - \int x\cdot\frac{1}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)+c$

Let $1+x^2 = t$
$2x\,dx = dt$
$\int \frac{dt}{t}$
$= \ln t + c$
$= \ln(1+x^2)+c$

Example-4
Qns Integrate $\int(\ln x)^2\,dx.$

Soln $\int(\ln x)^2\,dx$

$= \int 1\times(\ln x)^2\,dx$

$= \left(\int dx\right)\cdot(\ln x)^2 - \int\left(\int dx\right)\frac{d}{dx}(\ln x)^2\cdot dx$

$= x(\ln x)^2 - \int x\cdot\frac{2\ln x}{x}\,dx.$

$= x(\ln x)^2 - 2\int 1\times \ln x\,dx$

$= x(\ln x)^2 - 2\left[x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx\right]$

$= x(\ln x)^2 - 2x\ln x + 2\int dx$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

$= x\left[(\ln x)^2 - 2(\ln x) + 2\right] + c$

$\frac{d}{dx}(\ln x)^2$
$= 2\ln x \times \frac{d}{dx}(\ln x)$
$= 2\ln x \times \frac{1}{x}$
$= 2\frac{\ln x}{x}$

Note $\int e^x\{f(x)+f'(x)\}\,dx = e^x\cdot f(x)+c$

Example-5 Imp. Integrate $\int \sqrt{x^2+a^2}\, dx$.

Sol: Let $I = \int \sqrt{x^2+a^2}\, dx$

$$I = \int \underset{II}{\sqrt{x^2+a^2}} \times \underset{I}{1}\, dx.$$

$$I = \left(\int dx\right) \cdot \sqrt{x^2+a^2} - \int \left(\left(\int dx\right) \cdot \frac{d}{dx}\left(\sqrt{x^2+a^2}\right)\right) dx$$

$$I = x\sqrt{x^2+a^2} - \int x \times \frac{1}{2\sqrt{x^2+a^2}} \times 2x\, dx.$$

$$I = x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}\, dx$$

$$I = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\, dx$$

$$I = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2}{\sqrt{x^2+a^2}}\, dx + \int \frac{a^2}{\sqrt{x^2+a^2}}\, dx$$

$$I = x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2+a^2}}$$

$$\cancel{I = x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\, dx}$$

$$I = x\sqrt{x^2+a^2} - I + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\Rightarrow 2I = x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

F

$$\Rightarrow \boxed{\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \ln\left|x+\sqrt{x^2+a^2}\right| + c}$$

Example-6 Imp. Integrate $\int \sqrt{x^2-a^2}\, dx$

Sol: Let $I = \int \sqrt{x^2-a^2}\, dx$

$$I = \int \underset{II}{\sqrt{x^2-a^2}} \times \underset{I}{1}\, dx$$

$$I = \left(\int dx\right) \sqrt{x^2-a^2} - \int \left(\left(\int dx\right) \cdot \frac{d}{dx}\left(\sqrt{x^2-a^2}\right)\right) dx.$$

$$I = x\sqrt{x^2-a^2} - \int x \times \frac{1}{2\sqrt{x^2-a^2}} \times 2x\, dx.$$

$$I = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}dx - \int \frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$I = x\sqrt{x^2-a^2} - I - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\Rightarrow 2I = x\sqrt{x^2-a^2} - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + c.$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\Rightarrow \boxed{\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c}$$ F

Example. 7

Imp Integrate $\int\sqrt{a^2-x^2}\,dx$.

Sol: Let $I = \int\sqrt{a^2-x^2}\,dx$

$$I = \int \sqrt{a^2-x^2} \times 1\,dx$$

$$I = \left(\int dx\right)\sqrt{a^2-x^2} - \int\left(\int dx\right)\cdot\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)dx$$

$$I = x\sqrt{a^2-x^2} - \int x\cdot\frac{1}{2\sqrt{a^2-x^2}}\times(-2x)\,dx$$

$$I = x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$I = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\sin^{-1}\frac{x}{a} + c$$

$$I = x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\frac{x}{a} + c$$

$\Rightarrow 2I = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a} + c$

$\Rightarrow I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

$\therefore \boxed{\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$ F

Example-8
IMP

Integrate $\int e^{ax}\sin bx\,dx$

Soln let $I = \int \underset{I}{e^{ax}}\,\underset{II}{\sin bx}\,dx$

$I = \left(\int e^{ax}dx\right)\sin bx - \int\left(\left(\int e^{ax}dx\right)\times\frac{d}{dx}(\sin bx)\times dx\right.$

$I = \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a}\times\cos bx\times b\,dx.$

$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int \underset{I}{e^{ax}}\,\underset{II}{\cos bx}\,dx$

$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\left(\int e^{ax}dx\right)\cos bx - \int\left(\int e^{ax}dx\right)\times\frac{d}{dx}(\cos bx)dx\right]$

$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx - \int\frac{e^{ax}}{a}\times(-\sin bx)\times b\,dx\right.$

$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx\right]$

$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$

$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I + c$

$\Rightarrow I + \frac{b^2}{a^2}I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx + c$

$\Rightarrow \left(\frac{a^2+b^2}{a^2}\right)I = e^{ax}\left[\frac{\sin bx}{a} - \frac{b}{a^2}\cos bx\right] + c$

$\Rightarrow \left(\frac{a^2+b^2}{\cancel{a^2}}\right)I = e^{ax}\left[\frac{a\sin bx - b\cos bx}{\cancel{a^2}}\right] + c$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c$

$\therefore \boxed{\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c}$ F Q:6

Similarly $\boxed{\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c}$ F

(Prove by yourself).
(Similar to above)

Exercises

Evaluate the following

1. $\int (1+x)e^x \, dx$
2. $\int x^3 e^x \, dx$
3. $\int x \sin x \, dx$
4. $\int x^2 \sin x \, dx$
5. $\int x \cos^2 x \, dx$
6. $\int 2x \cos 3x \cos 2x \, dx$
7. $\int 2x^3 \cos x^2 \, dx$
8. $\int x \cos x \, dx$
9. $\int (\ln x)^3 \, dx$
10. $\int x^7 \ln x \, dx$
11. $\int \sec^{-1} x \, dx$
12. $\int x \sin^{-1} x \, dx$
13. $\int e^x \cos^2 x \, dx$.
14. $\int e^{2x} \sin x \, dx$
15. $\int \sqrt{7x^2+2} \, dx$
16. $\int e^x (\tan x + \ln \sec x) \, dx$.
17. Imp $\int \left[\dfrac{1}{\ln x} - \dfrac{1}{(\ln x)^2}\right] dx$.
18. $\int \sin(\ln x) \, dx$
19. $\int \dfrac{x e^x}{(1+x)^2} \, dx$
20. $\int \sqrt{x^2-8} \, dx$.