DIFFERENTIAL EQUATION

A differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables

Ex:- $\frac{dy}{dx} + xy = x^2$

$$\frac{d^3y}{dx^3} + x^4 \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$$

$$\frac{dy}{dx} + \sin x = \cos x$$

Ordinary differential equation:-

An ordinary differential equation is one which involves only one independent variable.

Ex:- $\frac{dy}{dx} + y\tan x = \sec x$

Partial differential equation:-

A partial differential eq^n is one which involves more than one independent variable.

Ex:- $\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial t} = 4$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

ORDER AND DEGREE OF DIFFERENTIAL EQ^n

ORDER :- A differential EQ^n is said to be of order 'n' if the order of the highest order derivatives is 'n'

Example Find the order of the differential Eq^n each of the followings

Q-1. $\frac{dy}{dx} + 3y = x$

Order is '1'

Q.2. $\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Order is '2'

Note ⇒ Here $\frac{dy}{dx}$ is 1st order

$\frac{d^2y}{dx^2}$ is 2nd order derivative

So order is highest order derivative present in the eq^n.

Q.3. $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^5 = x$

Here $\frac{d^3y}{dx^3}$ is 3rd order derivative

and $\frac{dy}{dx}$ is 1st order derivative

So highest is 3rd order

∴ order is '3'

Q.4 $\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

Here $\frac{d^4y}{dx^4}$ is the 4th order derivative which is highest

So order of the given differential eq^n is 4

DEGREE :- A differential eq^n is said to be degree 'n' if the power of the highest order derivative in the eq^n is 'n' after the eq^n free from fractions and Radicals.

Ex

Radicals :- $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc are called Radicals.

Ex:- (1) $\frac{d^2y}{dx^2} = \sqrt{1 + \frac{dy}{dx}}$ (Here square root is present in the eq^n which is a radical so first we have to remove square root from the eq^n)

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = 1 + \frac{dy}{dx}$

Now power of highest order derivative is '2'

$\therefore$ degree is '2'

Ex (2) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = x\frac{d^2y}{dx^2}$ (Here power $\frac{3}{2}$ is radical so we have to remove it first)

squaring both side

$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = x^2\left(\frac{d^2y}{dx^2}\right)^2$

Now power of highest order derivative is '2'

$\therefore$ degree is '2'

Ex. Determine the order and degree of each of the following differential eqⁿ.

(i) $y\sec^2x\,dx + \tan x\,dy = 0$

$\Rightarrow \tan x\,dy = -y\sec^2x\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{-y\sec^2x}{\tan x}$

Here only $\frac{dy}{dx}$ is present which is a 1st order derivative and its power is '1'

$\therefore$ order is '1'

degree is '1'

(ii) $\left(\frac{dy}{dx}\right)^4 + y^5 = \frac{d^3y}{dx^3}$

Here $\frac{d^3y}{dx^3}$ is present which is 3rd order derivative and its power is '1'

$\therefore$ order '3'

degree '1'

(iii) $m\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Here power $\frac{3}{2}$ is given which is a radical so we have to remove it first

squaring both side

$m^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

Now highest order derivative is $\frac{d^2y}{dx^2}$ and its power is 2.

$\therefore$ order 2.

(iv) $\left(\frac{dy}{dx}\right)^2 + \frac{1}{\frac{dy}{dx}} = 2$

Here fraction is given so we have to remove the fraction first.

multiplying $\frac{dy}{dx}$ both side, we have

$$\left(\frac{dy}{dx}\right)^3 + 1 = 2\frac{dy}{dx}$$

Here only $\frac{dy}{dx}$ is present in the eq^n but highest power is '3'

order 1
degree 3

FORMATION OF A DIFFERENTIAL EQ^n

To form a differential eq^n from a given Relation means finding a differential eq^n whose solution is the given relation.
If a relation containing 'n' arbitrary constant then we differentiate the given relation n times to obtain n more equations. using all these eq^n we eliminate the constants. the eq^n so obtained is the differential eq^n.

Ex:- 1 Form a differential eq^n $y = A\sin x$

sol^n $y = A\sin x$ —① (Here 'A' is a constant to eliminate 'A' we have to differentiate one time to find the diff. eq^n)

$\frac{dy}{dx} = A\cos x$ —②

dividing ② by ①

$$\frac{\frac{dy}{dx}}{y} = \frac{A\cos x}{A\sin x} \Rightarrow \frac{dy}{dx} = y\cot x \text{ (Ans)}$$

Ex-2 If $y = Ae^{x} + Be^{-x}$ form a differential equation

Sol^n Here A, B two constants are present to eliminate A, B we have to differentiate two times to find the differential eq^n.

$$y = Ae^{x} + Be^{-x} \quad \text{--- (1)}$$

$$\Rightarrow \frac{dy}{dx} = Ae^{x} - Be^{-x} \quad \text{--- (2)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^{x} + Be^{-x} \quad \text{--- (3)}$$

Subtracting eq^n (1) from eq^n (3), we have

$$\frac{d^2y}{dx^2} - y = 0 \quad \text{(Ans)}$$

Ex-3 Find the differential eq^n of

$$y = c\tan^{-1}x$$

Sol^n
$$y = c\tan^{-1}x \quad \text{--- (1)}$$

$$\frac{dy}{dx} = \frac{c}{1+x^2} \quad \text{--- (2)}$$

$$\frac{\frac{dy}{dx}}{y} = \frac{\frac{\cancel{c}}{1+x^2}}{\cancel{c}\tan^{-1}x}$$

$$\Rightarrow \frac{\frac{dy}{dx}}{y} = \frac{\tan^{-1}x}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = y\frac{\tan^{-1}x}{1+x^2} \quad \text{(Ans)}$$

SOLUTION OF A DIFFERENTIAL EQ^n :-

The relation between the variables is called a sol^n of a differential eq^n.

$\Rightarrow$ There are two types of solutions

(1) General sol^n :-

The sol^n which contains as many as arbitrary constants as the order of differential eq^n is called the general sol^n.

Ex:- $y = A\cos x + B\sin x$ is a general sol^n of differential eq^n. $\frac{d^2y}{dx^2} + y = 0$

(2) Particular solution :-

The solution obtained by giving particular values to the arbitrary constants in the general sol^n is called particular solution.

Ex:- $y = 3\cos x + 2\sin x$ is a particular solution of differential eq^n $\frac{d^2y}{dx^2} + y = 0$

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential eq^n of first order and first degree involves x, y and $\frac{dy}{dx}$, so it can be put in any one of the following form

$$\frac{dy}{dx} = f(x, y)$$

or $$f\left(x, y, \frac{dy}{dx}\right) = 0$$

SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFF. EQ^n

The solution of first order and first degree eq^n is possible only when it falls under the category of some standard form. some standard forms are

(1) Variable separable form

(2) Linear differentential eq^n.

VARIABLE SEPARABLE FORM

If the differential eq^n of the form

$f(x)\,dx + g(y)\,dy = 0$ we say that the variable are separable and such eq^n can be solved by integration term separately.

The solution is given by

$$\int f(x)\,dx + \int g(y)\,dy = C$$

Example Solve $\frac{dy}{dx} = x^2+2x+5$

Sol^n $\frac{dy}{dx} = x^2+2x+5$

$\Rightarrow dy = (x^2+2x+5)\,dx$

$\Rightarrow \int dy = \int (x^2+2x+5)\,dx$

$\Rightarrow y = \frac{x^3}{3} + \frac{2x^2}{2} + 5x + C$

$\Rightarrow y = \frac{x^3}{3} + x^2 + 5x + C$ (Ans)

Ex-2 Solve $\frac{dy}{dx} = \frac{2y}{x^2+1}$

Sol^n $\frac{dy}{dx} = \frac{2y}{x^2+1}$

$\Rightarrow \frac{dy}{2y} = \frac{dx}{x^2+1}$

$\Rightarrow \int \frac{dy}{2y} = \int \frac{dx}{x^2+1} + C$

$\Rightarrow \frac{1}{2} \log y = \tan^{-1}x + C$

$\Rightarrow \log y = 2\tan^{-1}x + C$ (Ans)

Ex-3 Solve $\frac{dy}{dx} = x\cos x$

Sol^n $\frac{dy}{dx} = x\cos x$

$\Rightarrow dy = x\cos x \, dx$

$\Rightarrow \int dy = \int x\cos x \, dx$

$\Rightarrow y = x.\int \cos x \, dx - \int\left\{\frac{d}{dx}(x)\int \cos x \, dx\right\}dx$

$\Rightarrow y = x\sin x - \int 1.\sin x \, dx$

$\Rightarrow y = x\sin x + \cos x + C$ (Ans)

Ex-4 Solve $\frac{dy}{dx} = \sqrt{1-y^2}$

Sol^n $\frac{dy}{dx} = \sqrt{1-y^2}$

$\Rightarrow \frac{dy}{\sqrt{1-y^2}} = dx$

$\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int dx + C$

$\Rightarrow \sin^{-1}y = x + C$ (Ans)

Ex-5

Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Sol^n $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$\Rightarrow \sec^2 y \tan x\, dy = -\sec^2 x \tan y\, dx$

$\Rightarrow \frac{\sec^2 y}{\tan y} dy = -\frac{\sec^2 x}{\tan x} dx$

$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = -\int \frac{\sec^2 x}{\tan x} dx + C$

Let $\tan y = u$

$\Rightarrow \sec^2 y = \frac{du}{dy}$

$\Rightarrow \sec^2 y\, dy = du$

Let $\tan x = v$

$\Rightarrow \sec^2 x = \frac{dv}{dx}$

$\Rightarrow \sec^2 x\, dx = dv$

$\Rightarrow \int \frac{du}{u} = \int -\frac{dv}{v} + C$

$\Rightarrow \log u = -\log v + C$

or $\log u = -\log v + \log c$

(C is a constant also $\log c$ is a constant so 'C' can be changed to $\log c$)

$\Rightarrow \log u = \log \frac{c}{v}$

$\Rightarrow u = \frac{c}{v}$

$\Rightarrow \tan y = \frac{c}{\tan x}$

$\Rightarrow \tan y \cdot \tan x = c$

Ex-6 solve $x\cos^2y\,dx = y\cos^2x\,dy$

Sol^n $x\cos^2y\,dx = y\cos^2x\,dy$

$$\Rightarrow \frac{y\,dy}{\cos^2y} = \frac{x\,dx}{\cos^2x}$$

$$\Rightarrow y\cdot\sec^2y\,dy = x\sec^2x\,dx$$

$$\Rightarrow \int y\cdot\sec^2y\,dy = \int x\cdot\sec^2x\,dx$$

$$\Rightarrow y\cdot\int\sec^2y\,dy - \int\left(\frac{d}{dy}(y)\cdot\int\sec^2y\,dy\right)dy$$

$$= x\cdot\int\sec^2x\cdot dx - \int\left(\frac{d}{dx}(x)\int\sec^2x\cdot dx\right)dx$$

$$\Rightarrow y\cdot\tan y - \int 1\cdot\tan y\,dy = x\cdot\tan x - \int 1\cdot\tan x\cdot dx$$

$$\Rightarrow y\tan y - \log|\sec y| = x\tan x - \log|\sec x| + c$$

(Ans)

Ex-7 solve $(1+y^2)\,dx + (1+x^2)\,dy = 0$

Sol^n $(1+y^2)\,dx + (1+x^2)\,dy = 0$

$$\Rightarrow (1+x^2)\,dy = -(1+y^2)\,dx$$

$$\Rightarrow \frac{dy}{1+y^2} = -\frac{dx}{1+x^2}$$

$$\Rightarrow \int\frac{dy}{1+y^2} = -\int\frac{dx}{1+x^2}$$

($\tan^{-1}c$ can be taken as constant)

$$\Rightarrow \tan^{-1}y = -\tan^{-1}x + \tan^{-1}c$$

$$\tan^{-1}y + \tan^{-1}x = \tan^{-1}c$$

$$\Rightarrow \tan^{-1}\frac{y+x}{1-yx} = \tan^{-1}c$$

$$\Rightarrow \frac{y+x}{1-yx} = c$$

(Ans)

Ex-8 Find particular solⁿ of

$\frac{dy}{dx} = \cos^2 y$, $y = \pi/4$ when $x = 0$

Solⁿ $\frac{dy}{dx} = \cos^2 y$

$\Rightarrow \frac{dy}{\cos^2 y} = dx$

$\Rightarrow \sec^2 y \, dy = dx$

$\Rightarrow \int \sec^2 y \, dy = \int dx + C$

$\Rightarrow \tan y = x + C \longrightarrow$ General solⁿ

putting $x = 0$, $y = \pi/4$ (Given)

$\Rightarrow \tan \frac{\pi}{4} = 0 + C$

$\Rightarrow 1 = 0 + C$

$\Rightarrow C = 1$

Hence $\tan y = x + 1$ is the particular solⁿ

Ex-9 Find particular solⁿ of ~~(1+x)y~~ dy

$(1+x) y \, dx + (1-y) x \, dy = 0$ given $y = 2$ at $x = 1$

Solⁿ $(1+x) y \, dx + (1-y) x \, dy = 0$

$\Rightarrow (1-y) x \, dy = -(1+x) y \, dx$

$\Rightarrow \frac{1-y}{y} dy = -\frac{(1+x)}{x} dx$

$\Rightarrow \left(\frac{1}{y} - 1\right) dy = -\left(\frac{1}{x} + 1\right) dx$

$\Rightarrow \int \left(\frac{1}{y} - 1\right) dy = -\int \left(\frac{1}{x} + 1\right) dx$

$\Rightarrow \log y - y = -(\log x + x) + C \longrightarrow$ General solⁿ

Putting $y = 2$, $x = 1$

$\Rightarrow \log 2 - 2 = -(\log 1 + 1) + C$ $(\log 1 = 0)$

$\Rightarrow \log 2 - 2 + 1 = C$

$\Rightarrow C = \log 2 - 1$

... $(\log x + x) + \log 2 - 1$ is the particular

LINEAR DIFFERENTIAL EQ^n

A differential eq^n is said to be linear if the dependent variable and its derivative occurring in the eq^n are of first degree only and are not multiplied together.

Ex:- $\frac{dy}{dx} + y = \sin x$

$$\frac{dy}{dx} + y \tan x = \sec x$$

General form of Linear diff. eq^n

$$\boxed{\frac{dy}{dx} + Py = Q}$$

Where P and Q are the functions of x only or constant.

This type of differential eq^n are solved when they are multiplied by a factor, which is called integrating factor (I.F)

$$I.F = e^{\int P dx}$$

Then the solution is

$$\boxed{y(I.F) = \int Q.(I.F)\, dx + C}$$

or $\boxed{\frac{dx}{dy} + Px = Q}$ Here P, Q are function of y only or constant

$$I.F = e^{\int P dy}$$

Solution is $\boxed{x.(I.F) = \int Q.(I.F)\, dy + C}$

Example-1 solve $(1+x^2)\frac{dy}{dx} + 2xy = x^3$

Solⁿ $(1+x^2)\frac{dy}{dx} + 2xy = x^3$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{x^3}{1+x^2}$$

By comparing with the General form

$$\frac{dy}{dx} + Py = Q$$

we have $P = \frac{2x}{1+x^2}$ $Q = \frac{x^3}{1+x^2}$

Integrating factor $(I \cdot F) = e^{\int P dx}$

$= e^{\int \frac{2x}{1+x^2} dx}$ taking $1+x^2 = t$

$\Rightarrow 2x = \frac{dt}{dx}$

$= e^{\int \frac{dt}{t}}$ $\Rightarrow 2x dx = dt$

$= e^{\ln t}$ $(\because e^{\ln x} = x)$

$= t$

$= 1+x^2$

Solution is

$$y \cdot (I \cdot F) = \int Q (I \cdot F)\, dx + C$$

$$\Rightarrow y \cdot (1+x^2) = \int \frac{x^3}{\cancel{1+x^2}} \cdot \cancel{(1+x^2)}\, dx + C$$

$$\Rightarrow y(1+x^2) = \frac{x^4}{4} + C \quad \text{(Ans)}$$

Ex-2 solve $\frac{dy}{dx} + y = e^{-x}$

Solution:- By comparing with the general eq^n of Linear differential eq^n $\frac{dy}{dx} + Py = Q$

Here $P = 1$ $Q = e^{-x}$

$I.F = e^{\int P dx}$

$= e^{\int 1 dx}$

$= e^{x}$

Solution is

$y.(I.F) = \int Q.(I.F) dx + c$

$\Rightarrow y.e^{x} = \int e^{-x}.e^{x} dx + c$

$\Rightarrow y e^{x} = \int e^{0} dx + c$

$\Rightarrow y.e^{x} = \int 1 dx + c$

$\Rightarrow y.e^{x} = x + c$ (Ans)

Ex-3 Solve $(1-x^2)\frac{dy}{dx} - xy = 1$

Sol^n $(1-x^2)\frac{dy}{dx} - xy = 1$

$\Rightarrow \frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2}$

Compairing with general eq^n: $\frac{dy}{dx} + Py = Q$

$P = -\frac{x}{1-x^2} \qquad Q = \frac{1}{1-x^2}$

$I.F = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx}$ taking $1-x^2 = t$

$\Rightarrow -2x = \frac{dt}{dx}$

$-x dx = \frac{dt}{2}$

$= e^{\int \frac{dt/2}{t}}$

$\Rightarrow e^{1/2 \log t}$

$= e^{\log \sqrt{t}}$

$= \sqrt{t}$

$= \sqrt{1-x^2}$

Sol^n is $y \cdot (I.F) = \int Q (I.F) dx + C$

$\Rightarrow y \cdot \sqrt{1-x^2} = \int \frac{1}{1-x^2} \cdot \sqrt{1-x^2} + C$

$\Rightarrow y\sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2}} dx + C$

$\Rightarrow y\sqrt{1-x^2} = \sin^{-1}x + C$ (Ans)

Ex-3 Solve $\frac{dy}{dx} + y \cot x = \cos x$

Sol^n By comparing with the general eq^n

$$\frac{dy}{dx} + Py = Q$$

Here $P = \cot x$ $\quad Q = \cos x$

$$I.f = e^{\int P \cdot dx}$$

$$= e^{\int \cot x \, dx}$$

$$e^{\ln \sin x} \qquad (e^{\ln x} = x)$$

$$= \sin x$$

Sol^n is $y \cdot (I.f) = \int Q \cdot (I.f) \, dx + c$

$$\Rightarrow y \sin x = \int \cos x \cdot \sin x \, dx + c$$

$$\Rightarrow y \sin x = \int t \cdot dt + c$$

taking $\sin x = t$

$\cos x = \frac{dt}{dx}$

$\cos x \, dx = dt$

$$\Rightarrow y \sin x = \frac{t^2}{2} + c$$

$$\Rightarrow y \sin x = \frac{\sin^2 x}{2} + c \quad \text{(Ans)}$$

Ex-4 $\frac{dy}{dx} + y\sec x = \tan x$

Here $P = \sec x$ $\quad Q = \tan x$

$$I.F. = e^{\int \sec x\, dx}$$

$$= e^{\log(\sec x + \tan x)}$$

$$= \sec x + \tan x$$

Solⁿ is $y(\sec x + \tan x) = \int \tan x \cdot (\sec x + \tan x)\, dx + c$

$$= \int (\sec x \cdot \tan x + \tan^2 x)\, dx + c$$

$$= \int (\sec x \cdot \tan x + \sec^2 x - 1)\, dx + c$$

$$= \sec x + \tan x - x + c$$

(Ans

Ex-5 Solve $(x+y+1)\frac{dy}{dx} = 1$

Solⁿ $(x+y+1)\frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y+1}$$

$$\Rightarrow \frac{dx}{dy} = x+y+1$$

$$\Rightarrow \frac{dx}{dy} - x = y+1$$

By comparing with the General form

$$\frac{dx}{dy} + Px = Q$$

$P = -1 \quad Q = y+1$

$I.F = e^{\int P\,dy}$

$= e^{\int -1\,dy}$

$= e^{-y}$

Solⁿ $x(I.F) = \int Q(I.F)\,dy + c$

$x \cdot e^{-y} = \int (y+1) \cdot e^{-y} + c$

$= (y+1) \int e^{-y}\,dy - \int \left(\frac{d}{dy}(y+1) \cdot \int e^{-y}dy\right) \cdot dy$

$= (y+1)(-e^{-y}) - \int 1 \cdot -e^{-y}\,dy$

$= -e^{-y}(y+1) - e^{-y} + c$

$= -e^{-y}(y+1+1) + c$

$= -e^{-y}(y+2) + c$ (Ans)