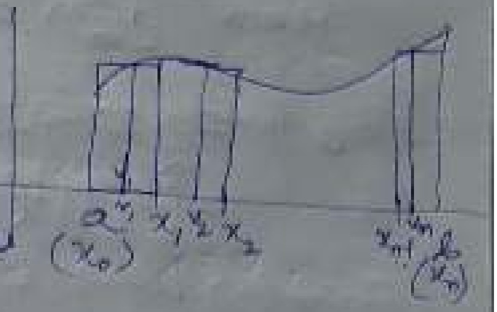
Definite Integral

Introduction

Integration can be considered as a process of summation, In this case the integral is called as definite Integral

Definition



Let $f(x)$ be a continuous function in $[a, b]$.

Divide $[a, b]$ into n sub-intervals of length $h_1, h_2, \ldots h_n$.

i.e. $h_1 = x_1 - x_0$, $h_2 = x_2 - x_1$, $\ldots\ldots$ $h_n = x_n - x_{n-1}$

Let v_r be any point in $[x_{r-1}, x_r]$ i.e.

$v_1 \in [x_0, x_1]$, $v_2 \in [x_1, x_2]$ $\ldots\ldots$, $v_n \in [x_{n-1}, x_n]$

Then the sum of area of the rectangles (as shown in figure) when $n \to \infty$ is defined as the definite Integral of $f(x)$ from a to b, denoted by

$$\int_a^b f(x)\,dx.$$

Here
a = lower limit of integration
b = upper limit of integration

Mathematically
$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[h_1 f(v_1) + h_2 f(v_2) + \cdots + h_n f(v_n)\right]}$$

Fundamental Theorem of Integral Calculus

If $f(x)$ is a continuous function in $[a, b]$ and $\int f(x)\,dx = \phi(x) + c$, Then

$$\boxed{\int_a^b f(x)\,dx = \phi(b) - \phi(a)}$$

Note - No arbitrary constants are used in definite integral.

Examble

1. Find $\int_1^2 x^3\,dx$

Ans

First find $\int x^3\,dx = \frac{x^4}{4} + c$

Here $f(x) = x^3$, $\varphi(x) = \frac{x^4}{4}$

By fundamental theorem

$$\int_1^2 x^3\,dx = \varphi(2) - \varphi(1)$$

$$= \frac{2^4}{4} - \frac{1^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

2. Find $\int_0^1 \frac{dx}{1+x^2}$

Ans

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

3. Find $\int_2^3 2x e^{x^2}\,dx$

Ans

$$\int_2^3 2x\,e^{x^2}\,dx$$

$$= \int_4^9 e^u\,du$$

$$= \left[e^u\right]_4^9$$

$$= e^9 - e^4 \text{ (Ans)}$$

Let $x^2 = u$

$\Rightarrow 2x\,dx = du$

when $x = 2$

$u = x^2 = 4$

& when $x = 3$

$u = x^2 = 9$

So lower limit change to 4 and upper limit changes to 9.

Properties of Definite Integral

1) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Explanation

Definite Integral is independent of variable.

E.g. $\int_2^3 x^2 dx = \int_2^3 u^2 du = \int_2^3 t^2 dt$ (Check it)

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Explanation

If limits of definite integral are interchanged then the value changes to its negative.

E.g. $\int_2^3 x\,dx = -\int_3^2 x\,dx$ (Check)

3) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where $a < c < b$.

Explanation

If we integrate $f(x)$ in $[a,b]$. and $c \in [a,b]$ such that $a < c < b$, then the above integral is same if we integrate $f(x)$ in $[a,c]$ and $[c,b]$ and then add them.

E.g.→ $\int_2^6 x\,dx = \int_2^4 x\,dx + \int_4^6 x\,dx$

Verification → $\int_2^6 x\,dx = \left[\frac{x^2}{2}\right]_2^6 = \frac{6^2}{2} - \frac{2^2}{2} = \frac{36}{2} - \frac{4}{2}$

$= 18 - 2 = 16$ —(1)

$$\int_2^4 x\,dx + \int_4^6 x\,dx = \left[\frac{x^2}{2}\right]_2^4 + \left[\frac{x^2}{2}\right]_4^6$$

$$= \left[\frac{4^2}{2} - \frac{2^2}{2}\right] + \left[\frac{6^2}{2} - \frac{4^2}{2}\right]$$

$$= \left[\frac{16}{2} - \frac{4}{2}\right] + \left[\frac{36}{2} - \frac{16}{2}\right]$$

$$= (8-2) + (18-8)$$

$$= 6 + 10 = 16 \quad \text{—— (2)}$$

From (1) and (2) $\int_2^6 x\,dx = \int_2^4 x\,dx + \int_4^6 x\,dx$

(verified)

4) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

E.g $\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right) dx$

verification

$$\int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2}$$

$$= -\left[\cos\frac{\pi}{2} - \cos 0\right] = -[0-1]$$

$$= 1 \quad \text{—— (1)}$$

$$\int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos x\,dx$$

$$= \left[\sin x\right]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1 \quad \text{—— (2)}$$

From (1) and (2)

$\pi/2$ $\pi/2$

5) (i) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If $f(x)$ is an odd function then

$$\int_{-a}^{a} f(x)\,dx = 0$$

E.g. By this formula without integration we can find the internal for odd function. ~~and we can~~

Even function $\boxed{f(x) = f(-x)}$ and odd function $\boxed{-f(x) = f(-x)}$

E.g.

$$\int_{-2}^{2} x^2\,dx = 2\int_{0}^{2} x^2\,dx$$

As $f(x) = x^2$ is an even function

$f(-x) = (-x)^2 = x^2$ So, $f(-x) = f(x)$

$\Rightarrow f(x)$ is an even function.

Similarly

$$\boxed{\int_{-\pi/2}^{\pi/2} \sin x\,dx = 0}$$

Reason

$f(x) = \sin x \Rightarrow f(-x) = \sin(-x) = -\sin x$

So $f(-x) = -f(x)$

$\Rightarrow f(x)$ is an odd function

(Property (3), (4) and (5) are very important and very often used in problems)

6) (i) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

if $f(2a-x) = f(x)$

(ii) $\int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$

7) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

Problems

Q.1. Find $\int_{-2}^{1} |x|\,dx$

Ans

$$|x| = \begin{cases} -x & x<0 \\ x & x \geq 0 \end{cases}$$

$|x|$ changes its definition at '0', so divide the interval into two parts $[-2,0)$ and $(0,1)$

Now $\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx$

$$= \int_{-2}^{0} -x\,dx + \int_{0}^{1} x\,dx$$

{Property (3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$}

{ when $-2 < x < 0$ i.e. $x<0$ then $|x| = -x$

when $0 < x < 1$ i.e. $0 < x$ then $|x| = x$ }

$$= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}$$

$= -\left[\frac{0^2}{2} - \frac{(-2)^2}{2}\right] + \left[\frac{1^2}{2} - \frac{0^2}{2}\right]$

$= -\left[0 - \frac{4}{2}\right] + \left[\frac{1}{2} - 0\right]$

$= 2 + \frac{1}{2} = \frac{3}{2}$

2. $\int_{-6}^{6} |x+2|\,dx = ?$

Ans

$\int_{-6}^{6} |x+2|\,dx = \int_{-4}^{8} |u|\,du$

$\left\{\begin{array}{l} \text{Let } u = x+2 \Rightarrow du = dx \\ \text{when } x=-6,\ u = -6+2 = -4 \\ \text{when } x = 6,\ u = 6+2 = 8 \end{array}\right.$

$= \int_{-4}^{0} |u|\,du + \int_{0}^{8} |u|\,du$ {prop of $\int$ (3)

$= \int_{-4}^{0} -u\,du + \int_{0}^{8} u\,du$

$= -\left[\frac{u^2}{2}\right]_{-4}^{0} + \left[\frac{u^2}{2}\right]_{0}^{8}$

$= -\frac{1}{2}\left[u^2\right]_{-4}^{0} + \frac{1}{2}\left[u^2\right]_{0}^{8}$

$= -\frac{1}{2}\left[0^2 - (-4)^2\right] + \frac{1}{2}\left[8^2 - 0\right]$

$= -\frac{1}{2}(-16) + \frac{1}{2}(64) = 8 + 32 = 40$ (Ans)

3) Find $\int_1^3 [x]\, dx$

Ans [x] is a function which changes its value at every integral point. So, we have to break the ~~integral~~ range into different integral ranges i.e. (1,3) can be breaked into (1,2) and (2,3).

$$\int_1^3 [x]\, dx = \int_1^2 [x]\, dx + \int_2^3 [x]\, dx$$

{applying Property (3)

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

$$= \int_1^2 1 \cdot dx + \int_2^3 2\, dx$$

{when $1 < x < 2$, $[x] = 1$; when $2 < x < 3$, $[x] = 2$

$$= [x]_1^2 + [2x]_2^3$$

{By def^n of $[x]$

$$= (2-1) + 2(3-2)$$

$$= 1 + 2 \times 1$$

$$= 1 + 2 = 3.$$

4) Evaluate $\int_0^{3/2} [2x]\, dx$

Ans

$$\int_0^{3/2} [2x]\, dx = \int_0^3 [u]\, \frac{du}{2}$$

$$= \frac{1}{2} \int_0^3 [u]\, du$$

{Put $u = 2x$, $du = 2dx$ $\Rightarrow dx = \frac{du}{2}$; when $x = 0$, $u = 2x = 0$; when $x = 3/2$, $u = 2x = 3$

$= \frac{1}{2}\left[\int_0^1 [u]\,du + \int_1^2 [u]\,du + \int_2^3 [u]\,du\right]$

$= \frac{1}{2}\left[\int_0^1 0\cdot du + \int_1^2 1\cdot du + \int_2^3 2\cdot du\right]$

$= \frac{1}{2}\left[0 + [u]_1^2 + 2[u]_2^3\right]$

$= \frac{1}{2}\left[0 + (2-1) + 2(3-2)\right]$

$= \frac{1}{2}\left[0 + 1 + 2\right] = \frac{3}{2}$ (Ans)

5) find $\int_{-1}^{1} \{|x| + [x]\}\,dx$

Ans

$\int_{-1}^{1} (|x| + [x])\,dx = \int_{-1}^{1} |x|\,dx + \int_{-1}^{1} [x]\,dx$

$= \int_{-1}^{0} |x|\,dx + \int_0^1 |x|\,dx + \int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx$

$= \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx + \int_{-1}^{0} (-1)\,dx + \int_0^1 0\,dx$

$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 - [x]_{-1}^{0} + 0$ $\left\{ \begin{array}{l}\text{By defn of} \\ |x| \text{ \& } [x]\end{array}\right.$

$= -\frac{1}{2}\left(0^2 - (-1)^2\right) + \frac{1}{2}\left(1^2 - 0\right) - \left(0 - (-1)\right)$

$= -\frac{1}{2}(-1) + \frac{1}{2}\cdot 1 - (1)$

$= \frac{1}{2} + \frac{1}{2} - 1 = 0$ (Ans)

6) Evalute $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Ans

In this type of problems we generally used property(4). And this type of problem can be solved by following technique.

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ —— (1)

$\left\{ \text{As } \int_0^a f(x)dx = \int_0^a f(a-x)dx \right.$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$$

{ In above x is replaced by $\frac{\pi}{2}-x$. As by property(4) there is no change in integral value }

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\Rightarrow \boxed{I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx} \quad \text{—— (2)}$$

From (1) and (2) if we add (1) + (2)

$$\Rightarrow I + I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Hence $\boxed{\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}}$

7) Evalute $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$

Ans

Let $I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$

$$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right) d\theta \quad \left\{ \text{As } \int_0^a f(x)dx = \int_0^a f(a-x)dx \right.$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \left(\log 2 - \log(1+\tan\theta)\right) d\theta$$

$$\left\{ \log\left(\frac{a}{b}\right) = \log a - \log b \right\}$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

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$$\Rightarrow I = \int_0^{\pi/4} \log 2 \, d\theta - \int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

$$\Rightarrow I = \log 2 \int_0^{\pi/4} d\theta - I \quad \left\{ \text{as } I = \int_0^{\pi/4} \log(1+\tan\theta) \right.$$

$$\Rightarrow 2I = \log 2 \left[\theta\right]_0^{\pi/4} = \log 2 \left(\frac{\pi}{4} - 0\right)$$

$$\Rightarrow 2I = \frac{\pi}{4} \log 2$$

$$\Rightarrow \boxed{I = \frac{\pi}{8} \log 2}$$

Hence $\int_0^{\pi/4} \log(1+\tan\theta) d\theta = \frac{\pi}{8} \log 2$ (Ans)

8) Evalute $\int_0^1 \frac{x}{x+1} dx$

Ans

$$\int_0^1 \frac{x}{x+1} dx = \int_0^1 \frac{x+1-1}{x+1} dx$$

$$= \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1} \right) dx$$

$$= \int_0^1 \left(1 - \frac{1}{x+1} \right) dx$$

$$= \left[x - \ln(x+1) \right]_0^1$$

$$= \left[\left(1 - \ln(1+1)\right) - \left(0 - \ln(0+1)\right) \right]$$

$$= 1 - \ln 2 - 0 + \ln 1$$

$$= 1 - \ln 2 - 0 + 0 = 1 - \ln 2$$

9) $\int_0^1 \frac{dx}{e^x+e^{-x}}$

Ans

$$\int_0^1 \frac{dx}{e^x+e^{-x}} = \int_0^1 \frac{dx}{e^x+\frac{1}{e^x}}$$

$$= \int_0^1 \frac{dx}{\frac{e^x \cdot e^x+1}{e^x}}$$

$$= \int_0^1 \frac{e^x\,dx}{e^{x+x}+1} = \int_0^1 \frac{e^x}{e^{2x}+1}\,dx$$

$$= \int_0^1 \frac{e^x}{(e^x)^2+1}\,dx$$

Let $t = e^x$

$\Rightarrow dt = e^x dx$

when $x=0$

$t = e^x = e^0 = 1$

when $x = 1$

$t = e^1 = e$

$$= \int_1^e \frac{dt}{t^2+1}$$

$$= \left[\tan^{-1} t\right]_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4} \text{ (Ans)}$$

10) $\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = ?$

Ans

$$\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_1^{\sqrt{2}}$$

$$= \sec^{-1}\sqrt{2} - \sec^{-1} 1 = \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} \text{ (Ans)}$$

11) Find $\int_0^4 \frac{1}{x+\sqrt{x}}\,dx$

Ans

$$\int_0^4 \frac{1}{x+\sqrt{x}}\,dx$$

~~Let $\sqrt{x} = t^2$~~
~~$\Rightarrow \frac{1}{2\sqrt{x}}dx = 2t\,dt$~~
~~when $x = 0$~~
~~$t =$~~

Let $\sqrt{x} = t$

Then $\frac{1}{2\sqrt{x}}\,dx = dt$

$\Rightarrow dx = 2\sqrt{x}\,dt = 2t\,dt$

when $x=0$, $t = \sqrt{x} = \sqrt{0} = 0$

When $x=4$, $t = \sqrt{4} = 2$

Now $\int_0^4 \frac{1}{x+\sqrt{x}}\,dx = \int_0^2 \frac{2t\,dt}{t^2+t}$

$$= 2\int_0^2 \frac{t\,dt}{t(1+t)} = 2\int_0^2 \frac{dt}{1+t}$$

$$= 2\left[\ln(1+t)\right]_0^2 = 2(\ln 3 - \ln 1)$$

$$= 2(\ln 3 - 0) = 2\ln 3$$

12) Evaluate $\int_0^{\pi/2} \frac{1}{1+\cot x}\,dx$

Ans

$$\int_0^{\pi/2} \frac{1}{1+\cot x}\,dx = \int_0^{\pi/2} \frac{1}{1+\cot\left(\frac{\pi}{2}-x\right)}\,dx \quad \left\{\text{Prop (4)}\right.$$

$$= \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} \frac{1}{1+\frac{1}{\cot x}}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{\frac{\cot x + 1}{\cot x}} dx = \int_0^{\pi/2} \frac{\cot x \, dx}{1+\cot x}$$

$$= \int_0^{\pi/2} \frac{\cot x + 1 - 1}{1+\cot x} dx$$

$$= \int_0^{\pi/2} \left(1 - \frac{1}{1+\cot x}\right) dx$$

$$= \int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{1}{\cot x} dx \quad \text{——(1)}$$

Let $I = \int_0^{\pi/2} \frac{1}{\cot x} dx$

Then from (1)

$$I = \int_0^{\pi/2} dx - I$$

$$\Rightarrow 2I = [x]_0^{\pi/2} = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Hence $\int_0^{\pi/2} \frac{1}{1+\cot x} dx = \frac{\pi}{4}$ (Ans)

13) Evalute $\int_0^{\pi/4} \log(1+\tan x) dx$

Ans

$$I = \int_0^{\pi/4} \log(1+\tan x) \, dx$$

$$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx$$

$\left\{ \text{using } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right.$

$$= \int_0^{\pi/4} \log\left(1+\frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x}\right)$$

(See Problem 7)

Refer problem - 7

$$\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2 \quad (\text{Ans})$$

Q. Prove that

$$\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log\cos x\,dx$$

$$= -\frac{\pi}{2}\log 2$$

Ans

$$\text{Let } I = \int_0^{\pi/2} \log(\sin x)\,dx \quad \text{——— (1)}$$

$$= \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx \quad \left\{\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right.$$

$$\Rightarrow I = \int_0^{\pi/2} \log(\cos x)\,dx \quad \text{——— (2)}$$

(1) + (2)

$$I + I = \int_0^{\pi/2} \log(\sin x)\,dx + \int_0^{\pi/2} \log(\cos x)\,dx$$

$$= \int_0^{\pi/2} \left\{\log(\sin x) + \log(\cos x)\right\}dx$$

$$= \int_0^{\pi/2} \log(\sin x\cdot\cos x)\,dx \quad \left\{\begin{array}{l}\log a + \log b \\ = \log ab\end{array}\right.$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int (\log\sin 2x - \log 2)\,dx$$

$\Rightarrow 2I = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$ ——(3)

Now $\int_0^{\pi/2} \log \sin 2x \, dx$

Put $t = 2x$ we have $dt = 2dx$

when $x = 0$, $t = 0$

when $x = \frac{\pi}{2}$, $t = \pi$

$= \int_0^{\pi} \log \sin t \, \frac{dt}{2}$

$= \frac{1}{2} \int_0^{\pi} \log(\sin t) \, dt$

Here $f(t) = \log \sin(t)$

$f(\pi - t) = \log \sin(\pi - t) = \log \sin(t) = f(t)$

Now by property $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ when $f(2a-x) = f(x)$

we have $\int_0^{\pi} \log(\sin t)dt = 2\int_0^{\pi/2} \log \sin t \, dt$

$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log(\sin t) \, dt$

$= \int_0^{\pi/2} \log(\sin t) \, dt$

$= \int_0^{\pi/2} \log(\sin x) \, dx$

Property (1) $\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$

$= I$ (from (1))

$\Rightarrow \boxed{I = \int_0^{\pi/2} \log \sin 2x \, dx}$ ——(4)

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From (3) and (4)

$$2I = I - \int_0^{\pi/2} \log 2 \, dx$$

$$\Rightarrow 2I = I - \log 2 \, [x]_0^{\pi/2}$$

$$\Rightarrow I = -\log 2 \left(\frac{\pi}{2} - 0\right)$$

Hence $\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log \cos x \, dx$

$$= -\frac{\pi}{2} \log 2 \quad \text{(Proved)}$$

Assignments

Evaluate the Integrals

1) $\int_0^2 [x^2]\,dx$

2) $\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$

3) $\int_0^{\pi/2} \frac{1}{1 + \tan x}\,dx$

4) $\int_0^1 \frac{\log(1+x)}{(1+x^2)}\,dx$

5) $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}}\,dx$

6) $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

7) $\int_0^{\pi/2} \log(\tan x)\,dx$

8) $\int_0^{\pi/2} \frac{dx}{1+\sin x}$

9) $\int_0^{\pi/2} \sin 2x \log(\tan x)\, dx$

10) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$