LECTURE NOTES

ON

CONTROL SYSTEM ENGINEERING

(Elective – C)

Name of the course: Diploma in Electrical Engineering.

(6^{th} Semester)

Notes Prepared by: RADHAKRUSHNA DEY.

Designation : LECTURER IN ELECTRICAL.

College : UTKALMANI GOPABANDHU INSTITUTE OF ENGINEERING, ROURKELA.

Signal Flow Graph

Introduction :

For a system with complex interrelations, the block diagram reduction procedure is complicated and often quite difficult to complete. An alternative method for determining the relationships between system variables has been developed by S. J. Mason.

Block diagrams and signal-flow graphs are closely related and accomplish the same purpose. But the signal flow graph has the advantage of more uniform notation. The signal flow graph is more convenient to draw and easier to visualize.

Signal flow graphs are an alternative to block diagrams. Unlike block diagrams which consists of blocks, signals, summing points, and takeoff points, a signal flow graph consists only of branches, which represents systems and nodes which represents signals.

Signal Flow Graph :

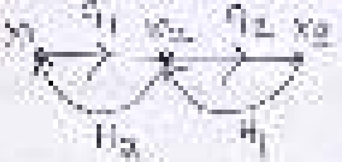
Node : A node of a signal flow graph is a point representing a system variable. It is denoted by a small dot (•) or a small circle (o). In fig x_1, x_2, x_3, x_4, x_5 are nodes.

Branch : A branch is a connection between two nodes of a network and represents the dependence of one node on the other.

Classification of node:

There are three types of nodes.

1. Input node (Source node) — A node with only outgoing branches.
2. Output node (Sink node) — A node with only incoming branches.
3. Chain node (Mixed node) — A node with both incoming and outgoing branches.
4. Dummy node — If the incoming and outgoing branches exist at the first and the last node representing input and output variables, then separate input and output nodes are created by adding branches with gain 1.



System without input and output node.

System with dummy nodes.

Path: A path is a traversal of connected unidirectional succession of branches along which no node is traversed more than once.

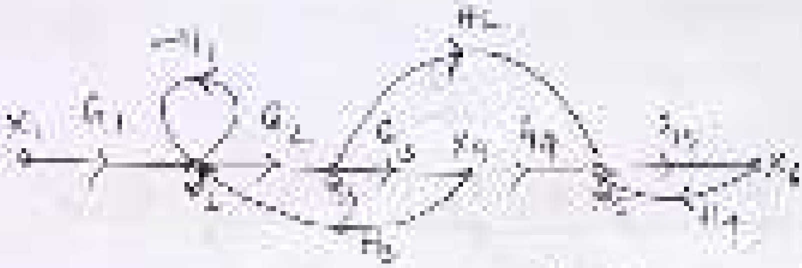
Forward path: A forward path is a path which starts at input node and ends at output node along the direction of the signal flow. no node should be traversed more than once.

Forward path Gain: The product of gains found by traversing a path from input to the output node of the signal flowgraph in the direction of signal flow.

Loop: A closed path in the direction of arrows which originates and terminates at the same node and along which no node is traversed more than once.

Self loop: A self loop is defined as a feedback path of only one branch which starts and terminates at the same node. example → H_2

Non touching Loops: Non touching loops are such loops having no path or branches in common.



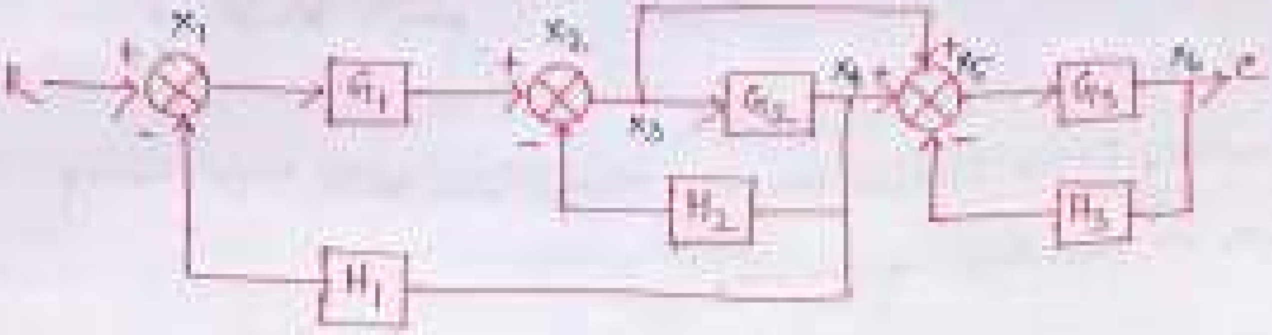
Loop-1 – $x_2 x_3 x_4 x_2$ Loop 2 $x_5 x_5 x_5$ are non-touching

Similarly Loop-1 – $x_2 x_2$ Loop 2 – $x_4 x_5 x_4$ are non-touching

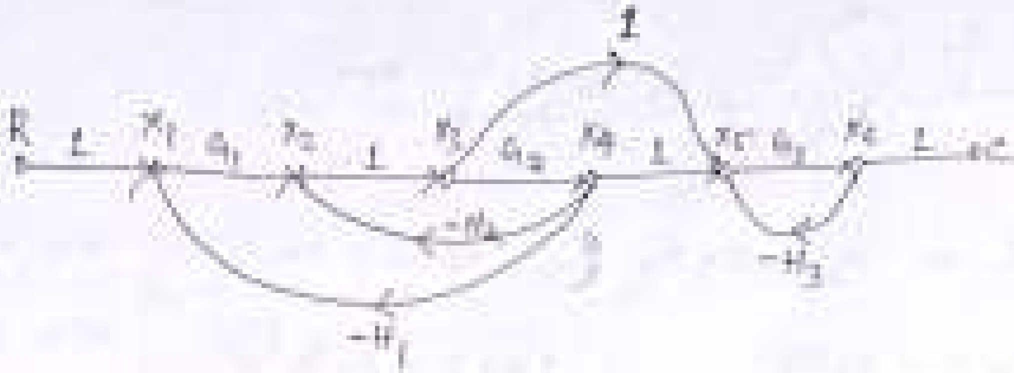
Loop gain: The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow without passing through any other node more than once.

Gain of Loop-1 → $G_2 G_3 H_3$; Gain of Loop-2 → $G_5 H_4$; Gain of Loops → $-H_1$

Qus: Convert the block-diagram to signal flow graph.



Ans:—



Mason's Gain Formula :

The relationship between an input variable and an output variable of a signal flowgraph is given by the net gain between the input and output nodes. This gain is called the over all gain of the system.

We can find this over all gain by Mason's Gain Formula.

$$T.F = \frac{C}{R} = \sum_{K=1}^{N} \frac{P_K \Delta_K}{\Delta}$$

Where N = no. of forward paths

$K = 1, 2, 3 \ldots N$.

P_K = Gain product of K^{th} forward path.

Δ = Determinant of signal flow graph.

Scanned by CamScanner

$\Delta = 1 - (\Sigma$ all loop gains$)$

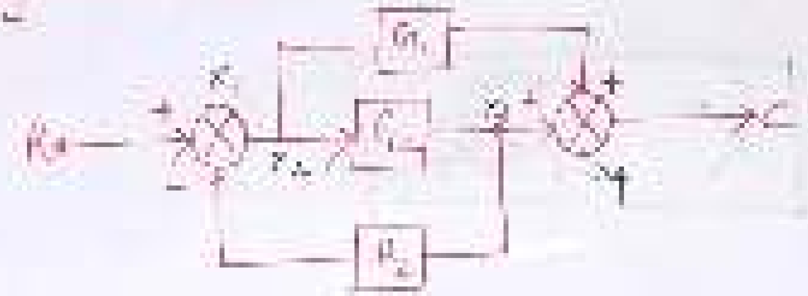
$+ (\Sigma$ products of gains of all possible of two nontouching loops$)$

$- (\Sigma$ products of " " " " three " loops$)$

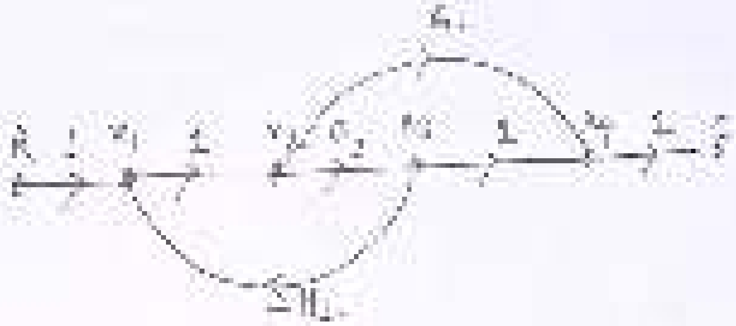
$+ \ldots\ldots$

Δ_K = It is the value of Δ not touching the Kth forward path gain.

Qn: Draw the signal flow graph and find the overall transfer function by Mason's gain formula.



Ans:- Step-1



Step-2 There are two forward paths

$P_1 = R \to x_1 \to x_2 \to x_3 \to x_4 \to C = 1 \times 1 \times G_2 \times 1 \times 1 = G_2$

$P_2 = R \to x_1 \to x_2 \to x_4 \to C = 1 \times 1 \times G_1 \times 1 = G_1$

Step-3 There is one loop here.

$L_1 = x_1 \to x_2 \to x_3 \to x_1 = 1 \times G_2 \times (-H_2) = -G_2 H_2$

Step-4 determinant of the SFG

$$\Delta = 1 - L_1 = 1 - (-G_2H_2) = 1 + G_2H_2$$

Step-5 Find the forward path factors det Δ_k

$$\Delta_1 = 1 - 0 = 1$$

$$\Delta_2 = 1 - 0 = 1$$

Step-6 By Mason's Gain Formula

$$\frac{C}{R} = \frac{P_1\Delta_1}{\Delta} + \frac{P_2\Delta_2}{\Delta}$$

$$= \frac{1}{1+G_2H_2}\left\{G_1 \times 1 + G_2 \times 1\right\}$$

$$\boxed{\frac{C}{R} = \frac{G_1+G_2}{1+G_2H_2}}$$ (Ans)

TIME RESPONSE ANALYSIS

Time response of a system is the output of the system as a function of time, when subjected to a given input.

Mainly time response is divided into two types

i) Transient response.

ii) Steady state response.

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

where $c(t)$ = total time response

$c_{tr}(t)$ = transient time response

$c_{ss}(t)$ = steady-state response.

Standard test inputs:-

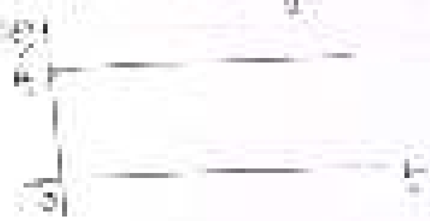
For design and analysis of a control systems, some test signals are used as inputs. The most commonly used inputs for this purpose are —

- step function
- Ramp function
- parabolic function
- Impulse function.

Step function:-

Mathematically the step function is given by

$$\left.\begin{array}{l} r(t) = 0 \text{ for } t < 0 \\ r(t) = A \text{ for } t \geq 0 \end{array}\right\}$$



If $k=1$ the function $r(t)=u(t)=1$

it is known as unit step function.

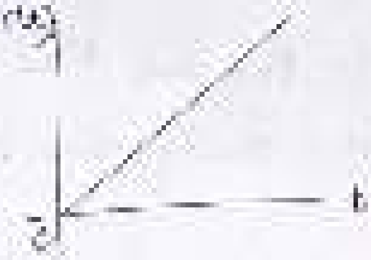
The Laplace transform of unit step function is given by

$$R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1\} = 1/s$$

$$\boxed{R(s) = 1/s}$$

Ramp function:—

The ramp function can be mathematically represented by



$$\left.\begin{array}{ll} r(t) = 0 & \text{for } t<0 \\ r(t) = kt & \text{for } t>0 \end{array}\right\}$$

Where k: slope of the line

if $k=1$, $r(t)=t$

Hence it is known as unit ramp function.

Laplace transform of unit ramp function

$$R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

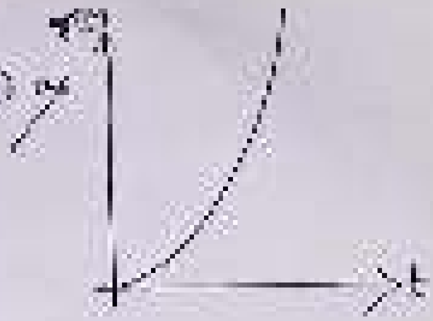
$$\boxed{R(s) = \frac{1}{s^2}}$$

- Ramp function is also called velocity function & note that

$$\boxed{\int(\text{unit-step function}) = \text{unit ramp function}}$$

Parabolic function

Mathematically this function can be represented as



$$r(t) = 0 \quad \text{for } t \le 0$$
$$r(t) = \frac{1}{2} t^2 \quad \text{for } t \ge 0$$

Laplace transform of unit parabolic function is

$$R(s) = \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3}$$

$$\boxed{R(s) = \frac{1}{s^3}}$$

∴ Parabolic function is also called acceleration function.

$\boxed{\int \text{unit ramp function} = \text{unit parabolic function}}$

Impulse Function

For an impulse function the duration (width) of the pulse approaching zero, the amplitude of the pulse approaching infinity, but the area of the pulse still remains unity.

Mathematically a unit impulse function is defined as

$$\delta(t) = 0 \quad \text{for } t \ne 0$$
$$\delta(t) = \infty \quad \text{for } t = 0$$

Since area under the impulse is unity $\boxed{\int_{-\infty}^{\infty} \delta(t)\, dt = 1}$

$\boxed{\int \text{impulse function} = \text{unit step function}}$

∴ $\int$ → integration
∴ D → Derivative

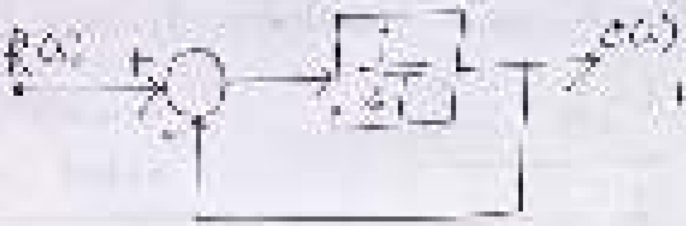
Impulse $\xrightarrow{\int}$ unit step $\xrightarrow{\int}$ unit ramp $\xrightarrow{\int}$ parabolic

Parabolic $\xrightarrow{D}$ unit ramp $\xrightarrow{D}$ unit step $\xrightarrow{D}$ Impulse

Time Response for a First Order System:-

First Order Control System:- A control system in which the highest power of 's' in the denominator of its transfer function is equal to '1'.

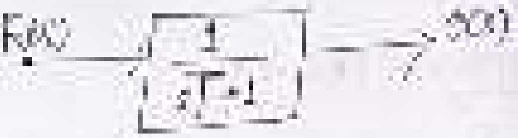
example:-



here $G(s) = \frac{1}{sT}$

$H(s) = 1$

transfer function of the system



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{1/sT}{1+(1/sT)(1)} = \frac{1}{sT+1}$$

Response of the first order System with unit step input

For a first order System $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$

$$C(s) = R(s) \cdot \frac{1}{sT+1} \quad \text{——— (eq}^n\text{ - 1)}$$

For an unit step input $r(t) = 1$ and $R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1\} = \frac{1}{s}$

putting the value of $R(s)$ in (eq^n 1)

$$C(s) = \frac{1}{s} \cdot \frac{1}{sT+1}$$

Expanding this into partial fractions gives:

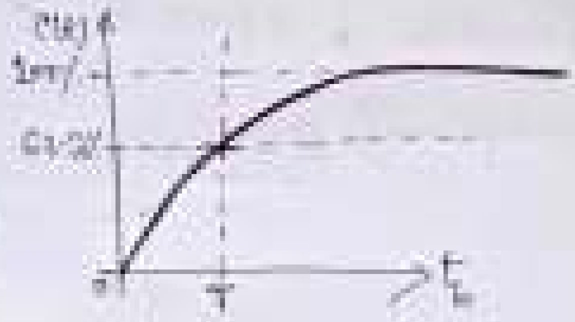
$$C(s) = \frac{1}{s} - \frac{T}{sT+1}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+(1/T)} \quad \text{—— (eq}^n\text{-2)}$$

Now taking inverse Laplace transform of eq^n-2,

$$\mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+(1/T)}\right\}$$

$$C(t) = 1 - e^{-t/T}$$

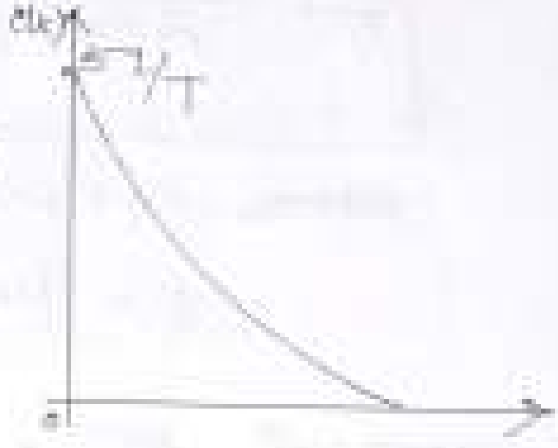


At $t=T$, $C(t) = 1 - e^{-1} = 1 - 0.368 = 0.632$.

Here T is time constant. It is defined as the time required for the output response to attain 63.2% of its final or steady state value.

Response of the first order system with unit impulse function

For unit impulse input



$$R(s) = 1.$$

$$C(s) = R(s) \cdot \frac{1}{sT+1} = 1 \cdot \frac{1}{sT+1}$$

$$C(s) = \frac{1}{s+\frac{1}{T}} \cdot \frac{1}{T}$$

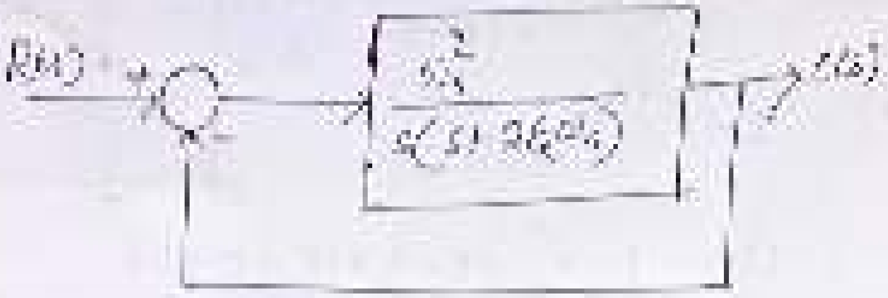
$$C(t) = \frac{1}{T} \cdot e^{-t/T}.$$

$$C(0) = \frac{1}{T} \cdot e^{0} = \frac{1}{T} \times 1 = \frac{1}{T}$$

Time Response for a second order System:

Second order System:- A control system in which the highest power of 's' in the denominator of the transfer function is equal to '2'.

example:- (1)



Here $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$, $H(s) = 1$

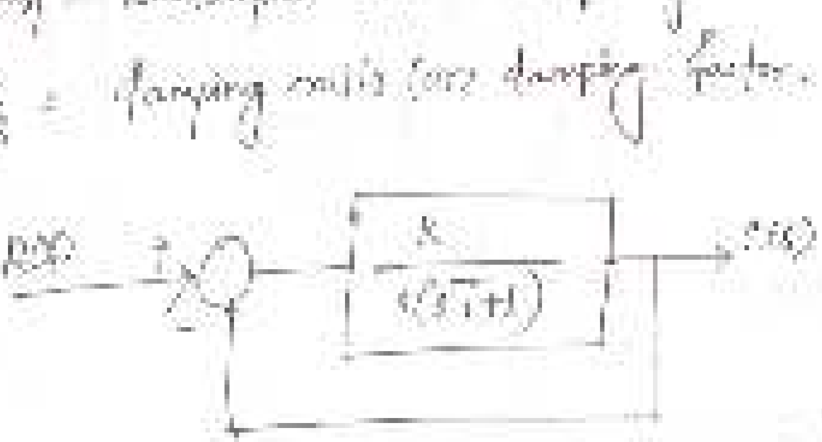
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\omega_n^2/s(s+2\xi\omega_n)}{1+\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}}$$ → standard form of 2nd order closed loop control system.

where ω_n = undamped natural frequency

ξ = damping ratio (or) damping factor.

example:- (2)



Here $G(s) = \dfrac{K}{s(sT+1)}$, $H(s) = 1$

The transfer function is $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K/\tau}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}}}$$

Comparing both transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K/\tau}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}}$$

From the above equalities we can find

$\omega_n = \sqrt{\frac{K}{\tau}}$ → undamped natural frequency | $\frac{1}{\zeta\omega_n}$ → time constant of the system

$\zeta = \frac{1}{2\omega_n\tau}$ → damping ratio.

In the standard form of second order system, the denominator polynomial $s^2 + 2\zeta\omega_n s + \omega_n^2$ is called "characteristic polynomial" and $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ is called "characteristic equation".

As we can see the characteristic eq^n is of a quadratic eq^n. solving this eq^n we can get the roots of this eq^n as

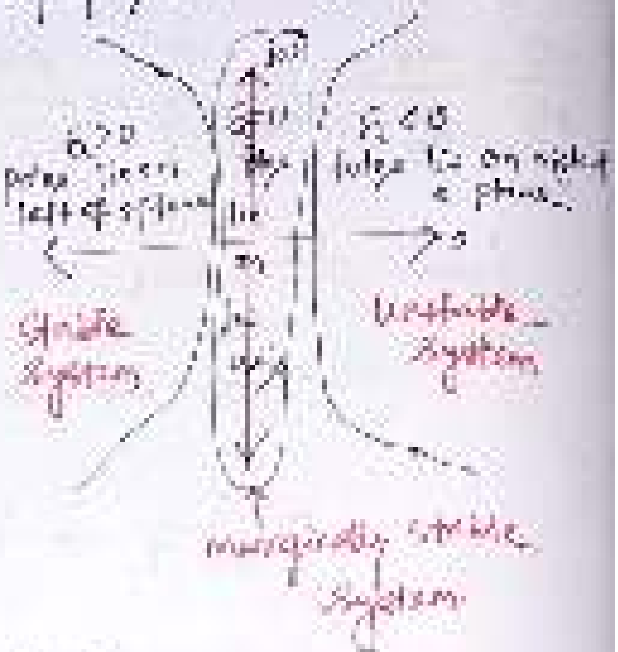
$$\left.\begin{aligned} s_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ s_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \end{aligned}\right\} \text{poles of the transfer function}$$

Case: we have four different cases

i) $\xi = 0$ (undamped) => In this case transient response doesn't die out.
system will be marginally stable

ii) $0 < \xi < 1$ (under damped) => closed loop poles s_1 and s_2 will be complex conjugate.

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$



Let $\omega_d = \omega_n\sqrt{1-\xi^2}$, hence

$$s_{1,2} = -\xi\omega_n \pm j\omega_d$$

iii) $\xi = 1$ (critically damped)

iv) $\xi > 1$ (over damped)

for $\xi < 0$ the system will be unstable.

i) Undamped case ($\xi = 0$)

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s)$$

For a unit step function as an input

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

for undamped case $\xi = 0$, hence

$$C(s) = \frac{\omega_n^2}{(s^2 + \omega_n^2)s}$$ (In partial fraction use the form $\frac{A}{s} + \frac{Bs + C}{(s^2 + \omega_n^2)}$)

Making partial fraction of $C(s)$ we can get

$$C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Scanned by CamScanner

Taking the inverse laplace transform, we can get

$$f(t) = 1 - \cos\omega_0 t \qquad \mathcal{L}(\cos\omega_0 t) = \frac{s}{s^2+\omega_0^2}$$

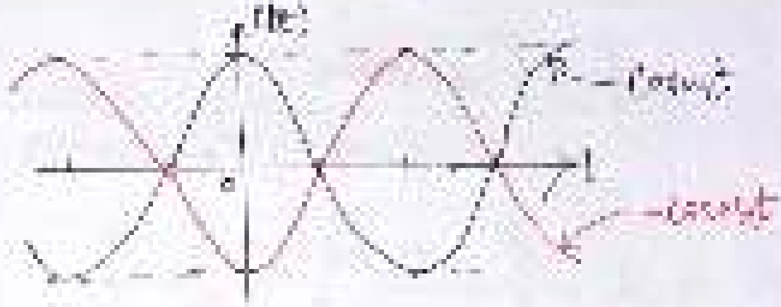
To draw the graph for $f(t) = 1 - \cos\omega_0 t$
we have to follow these steps

Step-1 draw for $\cos\omega_0 t$

Step-2 draw for $-\cos\omega_0 t$ (reflection)

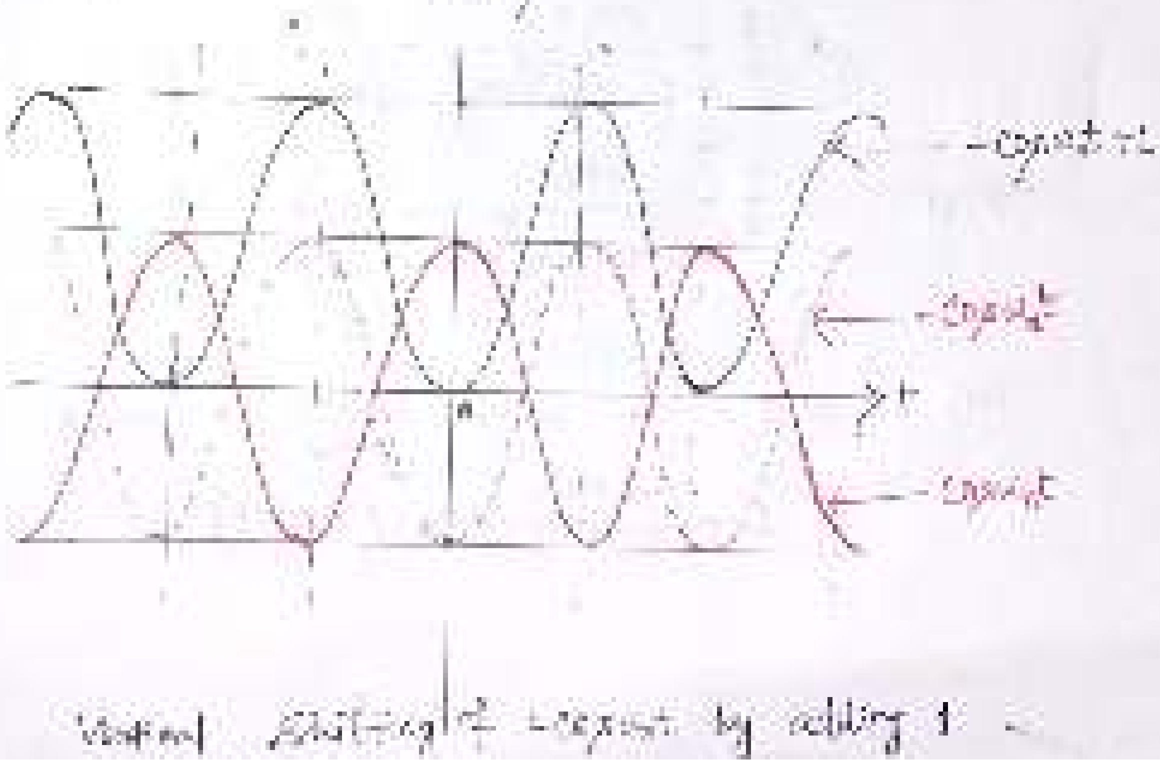
Step-3 draw for $-\cos\omega_0 t + 1$. (vertical shift)

Step-1 $\cos\omega_0 t$



Step-2 $-\cos\omega_0 t$ is the reflection of $\cos\omega_0 t$

Step-3 $-\cos\omega_0 t + 1$ is the upward shift of $-\cos\omega_0 t$ function to one unit



Vertical shifting of $-\cos\omega_0 t$ by adding 1.

ii) Underdamped case ($0<\xi<1$)

Output of the system $C(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \cdot R(s)$

For a unit step function $r(t)=1$ and $R(s)=\dfrac{1}{s}$

Therefore $C(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \cdot \dfrac{1}{s}$

After partial fraction we can get:

$$C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \left\{ \frac{s+2\xi\omega_n}{(s+\xi\omega_n - j\omega_d)(s+\xi\omega_n+j\omega_d)} \right\}$$

$$= \frac{1}{s} - \left\{ \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2-(j\omega_d)^2} \right\}$$

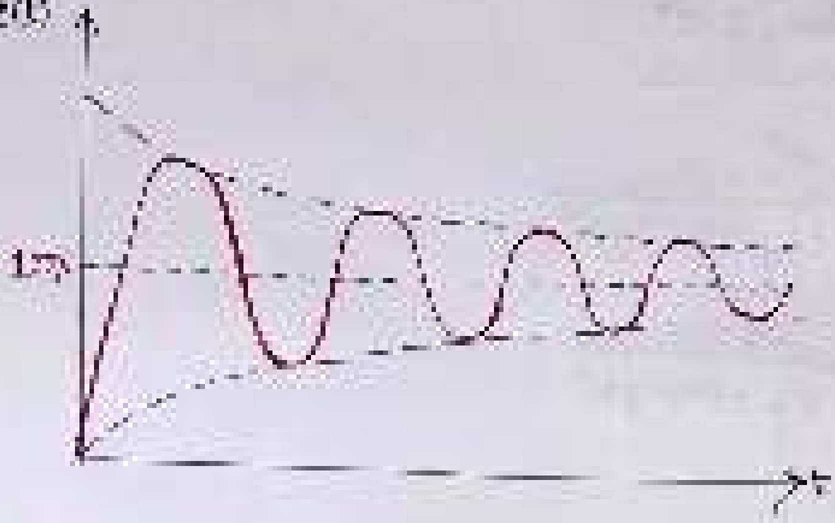
$$= \frac{1}{s} - \left\{ \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \right\}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\xi\omega_n t}\{\cos\omega_d t + \sin\omega_d t\}$$



Critically damped case ($\zeta = 1$)

If the two poles of $\frac{C(s)}{R(s)}$ are equal, the system is said to be critically damped case.

Putting $\zeta = 1$ in $C(s)$ eqn we can get

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\omega_n s + \omega_n^2)} R(s)$$

For unit step function as input $R(s) = 1/s$, hence

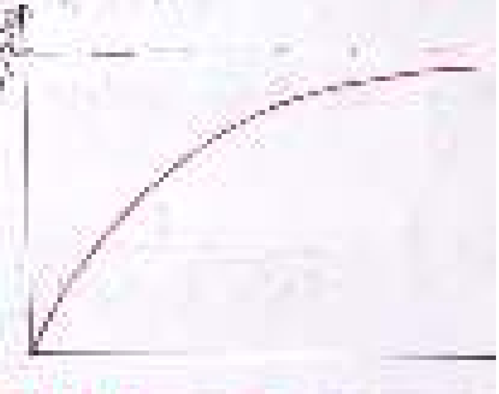
$$C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{\omega_n^2}{s\,(s+\omega_n)^2}$$

Partial fraction of $C(s)$: $\frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$

$$C(s) = \frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$L^{-1}(C(s)) = c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$



Scanned by CamScanner

Overdamped case ($\xi > 1$)

$$C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{s\left[s + \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]\left[s + \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]}$$

Let $\xi\omega_n + \omega_n\sqrt{\xi^2-1} = a$

$\xi\omega_n - \omega_n\sqrt{\xi^2-1} = b$

$$C(s) = \omega_n^2 \left\{ \frac{1}{s} \cdot \frac{1}{(s+a)(s+b)} \right\}$$

After partial fraction we can get

$$C(s) = \omega_n^2 \left\{ \frac{1}{s(ab)} + \frac{1}{(-a)(-a+b)(s+a)} + \frac{1}{(-b)(-b+a)(s+b)} \right\}$$

now $ab = (\xi\omega_n)^2 - \left(\omega_n\sqrt{\xi^2-1}\right)^2 = \omega_n^2$

$$-a(-a+b) = a^2 - ab = \left(\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)^2 - \omega_n^2 = 2\omega_n^2\sqrt{\xi^2-1}\left(\xi + \sqrt{\xi^2-1}\right)$$

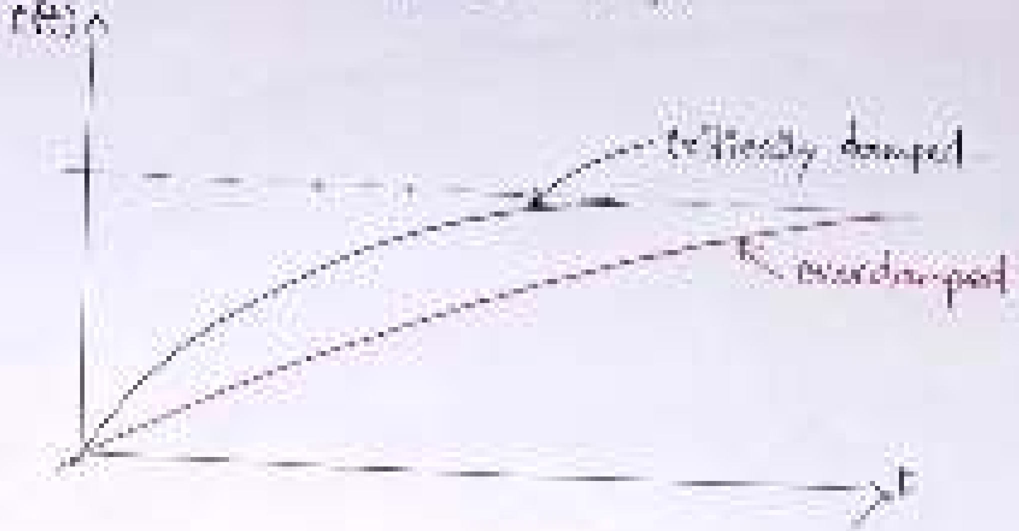
$$-b(-b+a) = b^2 - ab = \left(\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)^2 - \omega_n^2 = 2\omega_n^2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)$$

Putting all these values in $C(s)$ we get

$$C(s) = \frac{1}{s} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)\left[s + \left(\xi - \sqrt{\xi^2-1}\right)\omega_n\right]} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi + \sqrt{\xi^2-1}\right)\left[s + \left(\xi + \sqrt{\xi^2-1}\right)\omega_n\right]}$$

$$\mathcal{L}^{-1}\{C(s)\} = c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta+\sqrt{\zeta^2-1})}\, e^{-(\zeta+\sqrt{\zeta^2-1})t}$$

$$- \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})}\, e^{-(\zeta+\sqrt{\zeta^2-1})t}$$



Effect of ξ on pole locations in second-order systems.

Case.1 $0 < \xi < 1$

a) complex conjugate poles.

$$s_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$s_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$$

b) poles are located on the second and third quadrant.

c) An underdamped response.

Case.2 $\xi = 1$

a) Poles are real and equal. $s_1 = s_2 = -\xi\omega_n$.

b) Poles lie on the -ve σ axis.

c) A critically damped response.

Case-3 $\xi > 1$

a) poles are -ve, real, unequal

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2-1}$$
$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2-1}$$

b) poles lie on the -ve σ axis of imaginary plane.

c) An overdamped response.

Case-4 $\xi = 0$

a) The poles are complex with only imaginary part. They lie on the $j\omega$ axis. The poles are conjugates of each other.

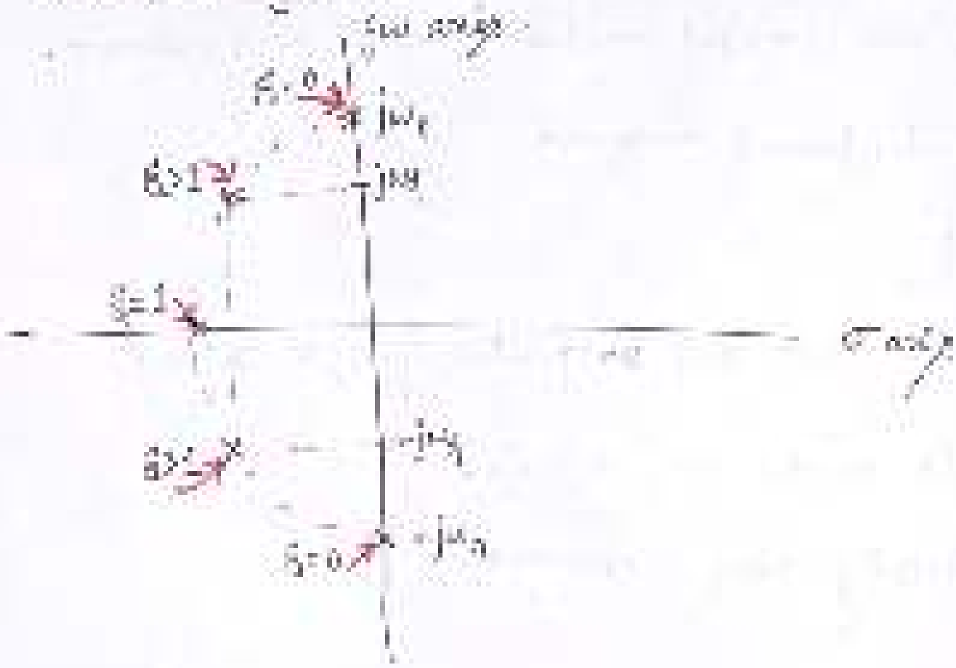
b) poles are given by $s_1 = -j\omega_n$
$s_2 = j\omega_n$

c) An undamped response.

Case-5 $\xi < 1$

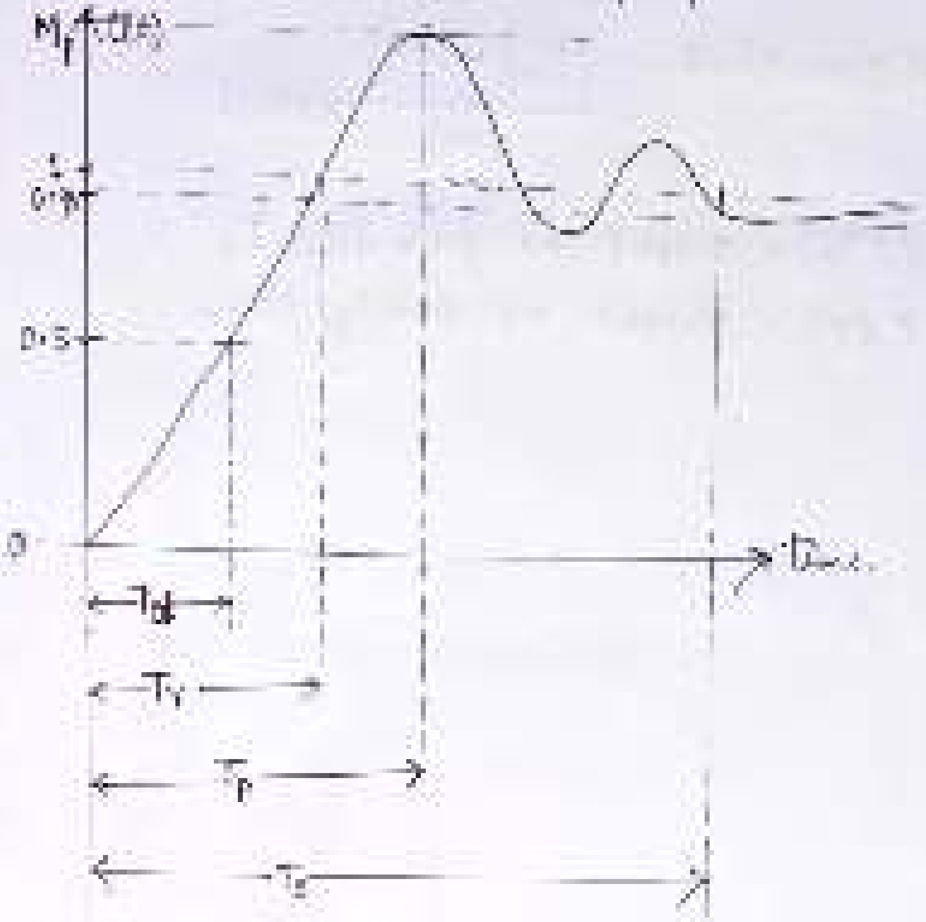
a) The poles lie on the right half of s-plane.

b) An unstable system.



Time Response Specifications

Before reaching steady state, all the control system undergoes transient state. This transient state can be analysed in terms of time response specifications.



Delay Time (T_d) :- The time for the system output response taken for the step input, to reach 50% of the final value is called delay time.

Rise Time (T_r) :- The time that the system output response takes for the step input, to reach 90% of its final value is called rise time.

Settling Time (T_s) :- This time is defined as the time required for the step response to settle down and stay within a specified tolerance of the final value.

Maximum Overshoot (or) Peak Overshoot :-

The ratio of the maximum value of the step excited output to the final output is called overshoot of the system.

$$\% \text{ overshoot} = \frac{c(T_p) - c(\infty)}{c(\infty)} \times 100\%$$

Where $c(T_p)$ = output at peak time T_p

$c(\infty)$ = output at steady state.

Derivation of Time response specifications

Rise Time (T_r) – we will consider the system as underdamped

Rise means $c(t) = 1$ at $t = T_r$.

We also know for an underdamped system

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t$$

we also know $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\text{So} \quad c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t$$

Putting $\sqrt{1-\xi^2} = \sin\phi$

we can get $\cos\phi = \xi$ and $\tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t\right]$$

we know: $\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$

Now: $$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) \quad \text{---- (eq}^n\text{-4)}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\therefore \boxed{c(t) = 1 - e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$

At rise time $t = T_r$, $c(t) = 1$.

then $$1 = 1 - e^{-\zeta\omega_n T_r} \sin\left(\omega_d T_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

So: $$e^{-\zeta\omega_n T_r} \sin\left(\omega_d T_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

Since $e^{-\zeta\omega_n T_r}$ can't be zero,

$$\sin\left(\omega_d T_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

$$\Rightarrow \omega_d T_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = 0$$

$$\Rightarrow T_r = -\frac{1}{\omega_d}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$= \frac{1}{\omega_d}\left[-\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

We know $\tan(\pi - \phi) = -\tan\phi$

Therefore $T_r = \dfrac{(\pi - \phi)}{\omega_d}$

$$\boxed{T_r = \dfrac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}}$$

Peak Time (T_p)

At peak time $\frac{d}{dt}(c(t)) = 0$

From eqn. A, we have $c(t) = 1 - \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$

Therefore, $\dfrac{d}{dt}c(t) = 0 + \dfrac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d t + \phi) - \dfrac{e^{-\xi\omega_n t} \cdot \omega_d \cos(\omega_d t + \phi)}{\sqrt{1-\xi^2}}$

Putting $\frac{d}{dt}c(t) = 0$ at $t = T_p$, we get

$$\frac{\xi\omega_n e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\omega_d T_p + \phi) - \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \cdot \omega_d \cos(\omega_d T_p + \phi) = 0$$

Since $\dfrac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \neq 0$

$$\xi\omega_n \sin(\omega_d T_p + \phi) - \omega_d \cos(\omega_d T_p + \phi) = 0$$

dividing $\cos(\omega_d T_p + \phi)$ to both sides of the eqn.

$$\xi\omega_n \tan(\omega_d T_p + \phi) - \omega_d = 0$$

$$\tan(\omega_d T_p + \phi) = \frac{\omega_d}{\xi \omega_n}$$

we know $\omega_d = \omega_n\sqrt{1-\xi^2}$

hence $\tan(\omega_d T_p + \phi) = \frac{\sqrt{1-\xi^2}}{\xi}$

we previously derived $\tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$

hence $\tan(\omega_d T_p + \phi) = \tan\phi$

we also know, $\tan(n\pi + \phi) = \tan\phi$.

hence $\tan(\omega_d T_p + \phi) = \tan(n\pi + \phi)$

$$\omega_d T_p + \phi = n\pi + \phi$$

where $n = 1, 2, 3$

$$\omega_d T_p = n\pi$$

$$T_p = \frac{n\pi}{\omega_d}$$

The first over-shoot is obtained for $n=1$ and the second overshoot is obtained for $n=2$.

Since the peak time corresponds to the first peak overshoot

$$n = 1$$

and $\omega_d T_p = \pi$

$$T_p = \frac{\pi}{\omega_d}$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

Settling Time (t_s)

As we know system never settles exactly at 100% hence some allowable tolerance band are defined like 2%, 5%. That means system will be at 98% and 95% during settling position.

$$c(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2}\, t + \phi)$$

$$0.98 = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n\sqrt{1-\xi^2}\, t + \phi)$$

When system gets settled sinusoidal term will be 1.

hence $0.98 = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n T_s}$

$$\Rightarrow \frac{e^{-\xi\omega_n T_s}}{\sqrt{1-\xi^2}} = 0.02$$

For low value of ξ, $\sqrt{1-\xi^2} = 1$

hence $e^{-\xi\omega_n T_s} = 0.02$

$$\ln\left(e^{-\xi\omega_n T_s}\right) = \ln(0.02)$$

$$-\xi\omega_n T_s = -4$$

$$T_s = 4/\xi\omega_n$$

$$T_s = 4 \times \frac{1}{\xi\omega_n}$$

we know Time constant $(T) = \frac{1}{\xi\omega_n}$

hence $\boxed{T_s = 4T}$ for 2% tolerance band.

Likewise we can derive $\boxed{T_s = 3T}$ for 5% tolerance band.

Peak over shoot (M_p)

The maximum over shoot occurs at the peak time or at

$$t = T_p = \frac{\pi}{\omega_d} \Rightarrow \pi = \omega_d T_p \text{ ----- } (eq^n - R)$$

Assuming that the final value of the output is unity,

$$M_p = c(t)\Big|_{t=T_p} - 1$$

$c(t)$

$$= \left[1 - \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\omega_d T_p + \phi)\right] - 1$$

Putting the value of $\omega_d T_p$ from $eq^n - R$

$$M_p = -\frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\pi + \phi)$$

We know $\sin(\pi+\phi) = -\sin\phi$.

$$M_p = \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin\phi.$$

Previously we have taken $\sin\phi = \sqrt{1-\xi^2}$

$$M_p = e^{-\xi\omega_n T_p}.$$

we know that $\omega_d = \omega_n\sqrt{1-\xi^2}$

$t_p = \pi/\omega_d$

Finally putting this values we can get.

$$M_p = e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

$$\boxed{M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

Types of Control Systems

The open loop transfer function of a system may be assumed of the form

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_z s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

Here, 'N' represents the no. of open loop poles at origin of the s-plane.

When N=0, the system is called 'type-zero' system.

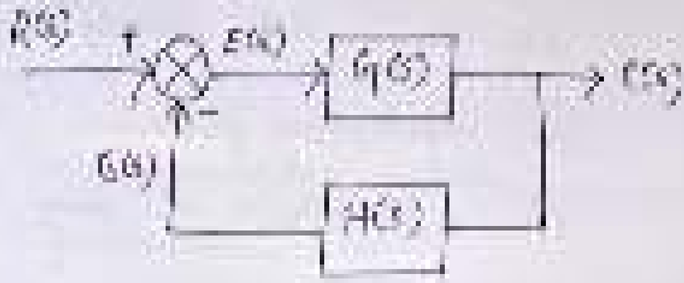
when N=1, the system is called "Type-one" system.

when N=2, the system is called 'type-two' system.

Scanned by CamScanner

Analysis of steady-state error

This is a closed-loop negative feedback system.



R(s) = applied input

C(s) = output

B(s) = feedback signal

E(s) = Error signal

from the figure 1

$$E(s) = R(s) - B(s)$$

$$B(s) = C(s)H(s)$$

$$E(s) = R(s) - C(s)H(s)$$

$$C(s) = E(s)G(s)$$

$$E(s) = R(s) - E(s)G(s)H(s)$$

then $E(s) + E(s)G(s)H(s) = R(s)$

$$E(s)\{1 + G(s)H(s)\} = R(s)$$

$$\boxed{E(s) = \frac{R(s)}{1 + G(s)H(s)}}$$

steady state error $(e_{ss}) = \lim_{s \to 0} sE(s)$

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)H(s)}}$$

Scanned by CamScanner

Steady state error depends on the following factors:

(i) type and magnitude of R(s)

(ii) open loop transfer function G(s)H(s).

(iii) the presence of any non-linearities.

Static error constants :—

Steady state error for a control system is found during the steady-state period. Hence steady state error is also known as static error.

Static position error constant (K_p)

position error constant K_p is associated with unit step input.

For unit step input

$$R(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1/s}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)H(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)H(s)}$$

Static position error constant (K_p) is defined by

$$\boxed{K_p = \lim_{s \to 0} G(s)H(s)}$$

$$\boxed{e_{ss} = \frac{1}{1 + K_p}}$$

Static velocity error constant (K_v)

Static velocity error constant K_v is associated with unit ramp input.

For unit ramp input

$$R(s) = \frac{1}{s^2}$$

Therefore, $$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \to 0} s + sG(s)H(s)}$$

The static velocity error constant $\boxed{(K_v) = \lim_{s \to 0} sG(s)H(s)}$

Therefore $\boxed{e_{ss} = \frac{1}{K_v}}$

Static Acceleration error constant (K_a)

For unit parabolic input, acceleration constant can be defined:

$$R(s) = \frac{1}{s^3}$$

Therefore $$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^3}$$

$$= \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s^2 G(s)H(s)}$$

The static acceleration error constant K_a is defined by

$$\boxed{K_a = \lim_{s \to 0} s^2 G(s)H(s)}$$

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

Examples:— 1) The closed loop transfer function of a second order unity feedback control system is given. Determine the type of damping in the system.

$$\frac{C(s)}{R(s)} = \frac{8}{s^2+2s+8}$$

Answer:— $$\frac{C(s)}{R(s)} = \frac{8}{s^2+2s+8} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Hence $\omega_n^2 = 8$

$\omega_n = \sqrt{8} = 2.82$ rad/s

$2\zeta\omega_n = 2$

$$\zeta = \frac{1}{2\omega_n} = \frac{2}{2\times 2.82} = 0.35$$

Since $\zeta < 1$, the given system is underdamped.

2) A unity feedback control system has an open loop transfer function $G(s) = \frac{5}{s(s+1)}$. Find the rise time, percentage overshoot, peak time and settling time for a step input of 10 units.

Solution:— $G(s) = \frac{5}{s(s+1)}$, $H(s) = 1$

The closed loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\frac{5}{s(s+1)}}{1+\frac{5}{s(s+1)}} = \frac{5}{s^2+s+5}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{5}{s^2+s+5}$$

Here $\omega_n^2 = 5$, $\omega_n = \sqrt{5} = 2.236$ rad/s and $2\zeta\omega_n = 1$

$$\zeta = \frac{1}{2\omega_n} = \frac{1}{2\times 2.236} = 0.223$$

Since, $\zeta < 1$, the given system is underdamped.

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= 2.236\times\sqrt{1-(0.223)^2} = 2.179 \text{ rad/s}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$= \tan^{-1}\frac{\sqrt{1-0.223^2}}{0.223} = 1.346 \text{ rad/sec}$$

Rise Time, $t_r = \frac{\pi-\phi}{\omega_d}$

$$= \frac{3.142-1.346}{2.179} = 0.825 \text{ sec}$$

Percentage peak overshoot for a unit step input

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

$$= e^{-0.223\pi/\sqrt{1-0.223^2}} \times 100$$

$$= 0.478 \times 100 = 47.8\%$$

Per unit peak overshoot for a unit step input = 0.4478

For an input of 10 units, the peak overshoot is

$$0.4478 \times 10 = 4.78$$

peak time, $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.124} = 1.479$ sec.

Time constant, $T = \frac{1}{\xi \omega_n} = \frac{1}{0.223 \times 2.236} = 2$ sec.

For 5% error, the settling time is

$$T_s = 3T = 3 \times 2 = 6 \text{ sec}$$

For 2% error, the settling time is

$$T_s = 4T = 4 \times 2 = 8 \text{ sec}$$

3) For unity feedback system having an open-loop transfer function

$$G(s) = \frac{K(s+2)}{s(s^3+7s^2+12s)}$$

Determine:

a) type of system b) error constants K_p, K_v, K_a

c) steady-state error for unit parabolic input.

Answer:— $H(s) = 1$

$$G(s)H(s) = \frac{K(s+2)}{s^2(s^2+7s+12)}$$

a) Since $G(s)H(s)$ has two poles at the origin of the s-plane so it is a type-2 system.

b) $K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K(s+2)/12}{s^2(\frac{s^2}{12}+\frac{7s}{12}+\frac{12}{12})} = \infty$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim \frac{K(s+2)/12}{s\left(\frac{s^2}{12} + \frac{7s}{12} + 1\right)} = \infty .$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{\frac{K}{12}(s+2)}{\frac{s^2}{12} + \frac{7s}{12} + 1} = \frac{K}{12}$$

(ii) Steady state error : $e_{ss} = \lim_{s\to 0} \dfrac{sR(s)}{1+G(s)H(s)}$

for unit parabolic input $R(s) = 1/s^3$.

$$e_{ss} = \lim_{s\to 0} \frac{1/s^2}{1 + \frac{K(s+2)}{s^2(s+4)(s+3)}}$$

$$= \lim_{s\to 0} \frac{(s+4)(s+3)}{s^2(s+4)(s+3) + K(s+2)} = \frac{12}{2K} = 6/K .$$

Methods to improve time response

It is necessary for a control system to meet certain specifications regarding its performance. System performance can be improved by using any of the following control methods.

i) Proportional Control

ii) Proportional plus Derivative Control

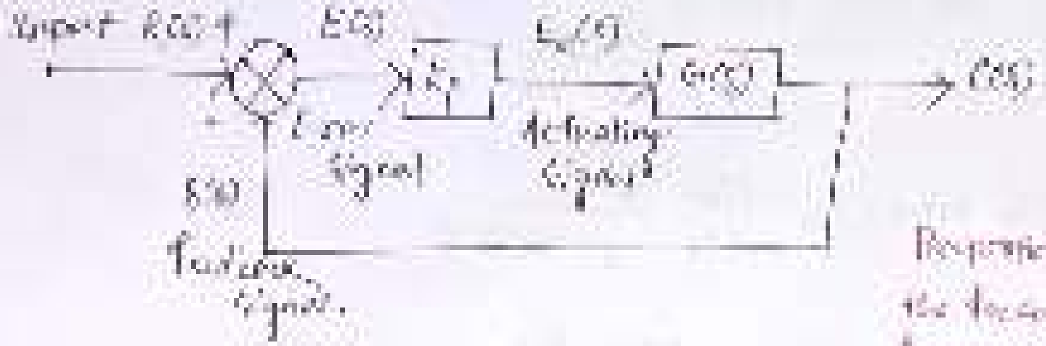
iii) Proportional plus Integral control

iv) Proportional plus Integral plus Derivative control.

A controller is a device which is introduced in feedback or forward path of a system, to control the steady-state and transient time according to the requirements.

Proportional Controller (P controller)

The proportional controller is a device that produces a control signal which is proportional to the input error signal.



from this figure

$$e_a(t) = K_p e(t)$$

$$E_a(s) = K_p E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p}$$

Proportional controller changes the forward path gain. If forward path gain is increased the peak overshoot increases but steady state error is reduced.

Proportional plus Derivative Controller (PD controller)

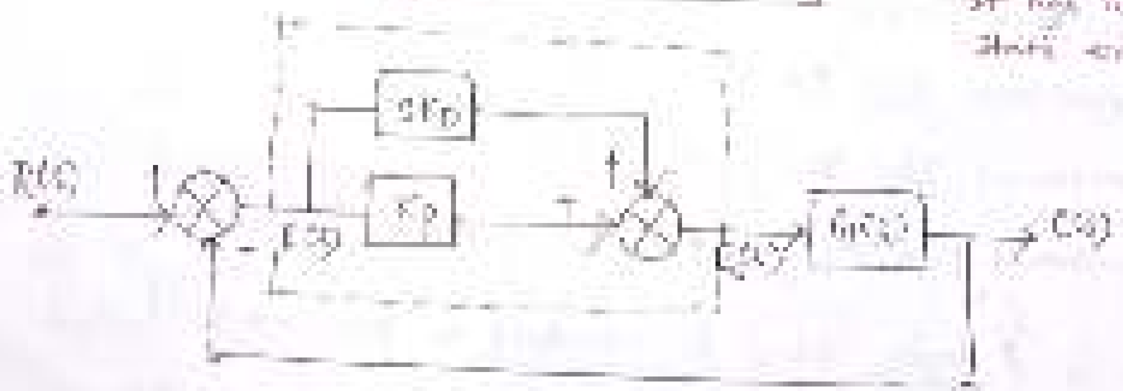
In this controller, the actuating signal $E_a(s)$ is proportional to the error signal $E(s)$ and also proportional to derivative of the error signal. Thus the actuating signal will be

$$e_a(t) = K_p e(t) + K_D \frac{d}{dt} e(t)$$

$$E_a(s) = K_p E(s) + K_D s E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p + K_D s}$$

By using this type controller overshoot can be reduced but derivative action (?) it has no effect on steady state error.



Proportional plus Integral Controller (PI controller)

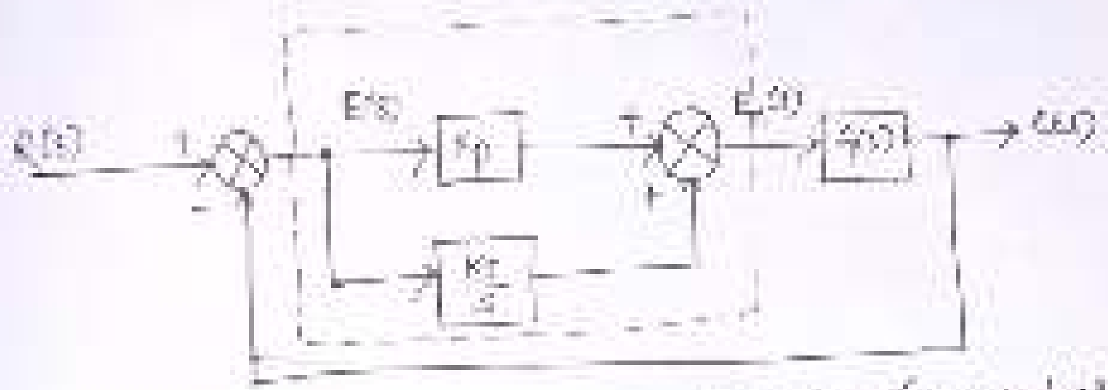
In this controller the actuating signal consists of proportional error signal adding to the integral of error signal.

$$e_a(t) = K_p e(t) + K_I \int e(t)\,dt$$

$$E_a(s) = K_p E(s) + \frac{K_I}{s} E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p + \frac{K_I}{s}}$$

By using this type controller we can improve the order of the system by one, which results in the reduction of the steady state error but the system stability may be hampered.



Proportional plus Integral plus Derivative Controller (PID Controller)

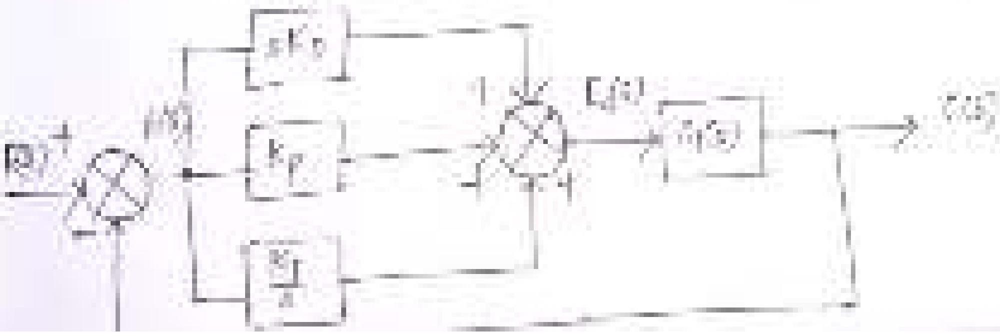
In this controller the actuating signal is proportional to error signal, to integral of error signal, to derivative of error signal.

$$e_a(t) = K_p e(t) + K_I \int e(t)\,dt + K_D \frac{d}{dt} e(t)$$

$$E_a(s) = K_p E(s) + \frac{K_I}{s} E(s) + sK_D E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p + sK_D + \frac{K_I}{s}}$$

The proportional controller stability, though, but produces a steady state error. The integral controller reduces the steady state error. The derivative controller reduces the rate of change of error. The main advantages of PID controller are higher stability, no offset and reduced overshoot.



Effect of adding poles and zeros to transfer functions

Addition of poles and zeros and/or cancellation of undesirable poles and zeros of the transfer function often are necessary to achieve satisfactory time-domain performance of control systems.

a) Addition of a pole to the forward path Transfer function.
The addition of a pole to the forward path transfer function increases the order of the system, increases the overshoot and reduces the stability. It also increases the rise time of the step response.

b) Addition of a pole to the closed-loop Transfer function.
The addition of a pole to the closed loop transfer function increases the rise time and decreases the over shoot.

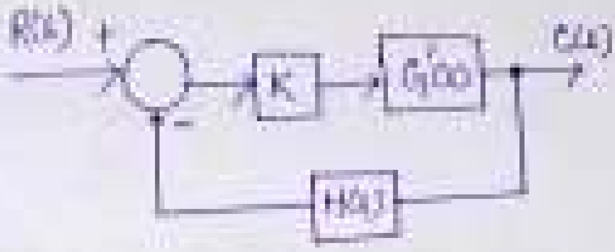
c) Addition of a zero to the closed loop transfer function.
Addition of a zero to the closed loop transfer function increases the maximum overshoot, decreases the rise time and increases the ... for step response.

d) Addition of a zero to the forward path transfer function
When the zero added to the forward path transfer function is very far away from the imaginary axis, the damping is very low — the overshoot over shoot will be large. The overshoot is reduced and damping improved when the zero moves to the right. Again, when the zero moves closer to the origin, the overshoot increases but damping improves.

Root Locus Analysis

Introduction:-

Consider a closed loop system



The transfer function of the forward path is $G(s) = KG'(s)$

Where K is the gain of the system and is the variable parameter. The characteristic eqn of the system is given by

$$1 + G(s)H(s) = 0$$

$$1 + KG'(s)H(s) = 0$$

The roots of the characteristic eqn are nothing but the closed loop poles, are dependent upon the value of K. If the gain K is varied from $-\infty$ to $+\infty$, a separate set of locations of the roots of the characteristic eqn will be obtained. The root locus is obtained by joining all such locations.

Root locus can be defined as the path of the closed loop poles traced out in the s-plane as the system open loop gain K is varied from $-\infty$ to $+\infty$. The variation of K will be assumed to be from 0 to $+\infty$ and the plot is known as "root locus".

Construction of root loci :-

(a) Starting points of root loci

Let the open-loop transfer function of a system may be written as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots(s+z_m)}{(s+P_1)(s+P_2)(s+P_3)\cdots(s+P_n)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+P_j)}$$

Therefore the characteristic eqn is

$$1 + G(s)H(s) = 0$$

or $$\prod_{j=1}^{n}(s+P_j) + K\prod_{i=1}^{m}(s+Z_i) = 0 \quad \text{—— eqn (2)}$$

When $K=0$, Eqn (2) has roots at $s = -P_j$ where $j = 1, 2, \dots n$

These are the open loop poles of the system. therefore the root locus branches start at open loop poles.

(iii) Terminating points of root loci

the characteristic eqn may also be written as

$$\frac{1}{K}\prod_{j=1}^{n}(s+P_j) + \prod_{i=1}^{m}(s+Z_i) = 0$$

When $K \to \infty$ $$\frac{1}{K}\prod_{j=1}^{n}(s+P_j) = 0$$

and $$\prod_{i=1}^{m}(s+Z_i) = 0 \quad \text{—— eqn (3)}$$

Eqn (3) shows that roots are located at $s = -Z_i$, where $i = 1, 2, \dots m$

These are the given zeros of the system. Therefore, the root locus branches terminate on end on the open loop zeros.

It is to be noted that for any given $G(s)H(s)$, if the number of finite zeros m is less than the no. of finite poles n, then $(n-m)$ zeros lie at infinity and $(n-m)$ branches of the root locus terminate on these zeros.

* Number of root loci (N)

The no. of separate loci N is given by

$N = m$ if $m > n$

$N = n$ if $n > m$

where m = no. of zeros and n = no. of poles.

* Existance of root loci on the real axis.

A point on the real axis lies on the root locus if and only if the sum of the open-loop poles and zeros, on the real axis, to the right of the point concerned is an odd number.

* Asymptotes and centroid

If the number of finite zeros, m, is less than the number of finite poles n, then (n-m) branches of the root locus must end at zeros at infinity. The branches which are approaching infinity, travel along the straight line called "asymptotes" of the root locus.

No. of asymptotes $= (n-m)$

All the asymptotes intersect the real axis at a common point called the centre of asymptotes or "centroid"

$$\text{centroid}(\sigma_A) = \frac{\text{Sum of real parts of poles of } G(s)H(s) - \text{Sum of real parts of zeros of } G(s)H(s)}{\text{Number of poles} - \text{Number of zeros}}$$

→ The angles between the asymptotes and the positive real axis are called "asymptotic angles"

$$\boxed{\phi_q = \frac{(2q+1)180^\circ}{n-m}}$$ where $q = 0, 1, 2, \ldots (n-m-1)$

Break Points

There are two types of break points.

(i) Breakaway point (ii) Break-in point

Breakaway point is defined as the point at which root locus comes out of the real axis and moves into the complex plane.

Break-in point is defined as a point at which root locus enters the real axis.

Procedure to find break points:

→ Write the characteristic eqn of the system

→ From the characteristic eqn, separate terms involving K and terms involving s and write the value of K in terms of 's'.

→ Differentiate above eqn with respect to 's' and equate it to zero, that is, $\frac{dK}{ds} = 0$.

→ The roots of the eqn $\frac{dK}{ds} = 0$ gives the break points.

→ In order to decide breakaway point or break-in point we determine $\frac{d^2K}{ds^2}$ and we substitute the break point value in $\frac{d^2K}{ds^2}$

if $\frac{d^2K}{ds^2} < 0$, then that point is breakaway point.

if $\frac{d^2K}{ds^2} > 0$, then that point is break-in point.

(4) Angle of Departure at a complex pole

The angle at which a root locus branch leaves a complex pole in the s-plane is called the 'angle of departure'. It is denoted by ϕ_d

$$\boxed{\phi_d = 180° - \phi}$$

where $\phi = \sum \phi_p - \sum \phi_z$

$\sum \phi_p$ = Sum of angles subtended by the phasors drawn from the poles to the pole (at which ϕ_d is to be calculated) from other poles.

$\sum \phi_z$ = Sum of angles subtended by the phasors drawn to the pole from all the zeros

(8) Imaginary-axis Crossing point :-

The point of intersection of the root locus branches with the $j\omega$ (imaginary) axis and the critical value of K, may be obtained by putting $s = j\omega$ in the characteristic equation and equating the real part and imaginary part to zero, and solving for ω and K. The value of ω is the intersection point on the imaginary axis and K at the intersection point, that is the marginal value of K.

Q:— For a unity feedback system, the open-loop transfer function is given by $G(s) = \frac{K}{s(s+2)(s^2+6s+25)}$

Sketch the root locus for $0 \le K \le \infty$

Ans:— $G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+25)}$

Step-1 For the $G(s)H(s)$, $s_1 = 0$, $s_2 = -2$

$s^2 + 6s + 25 = 0$

$s_{3,4} = \frac{-6 \pm \sqrt{36-100}}{2} = -3 \pm j4$

Therefore $s_3 = -3 + j4$, $s_4 = -3 - j4$.

Number of open loop poles $n = 4$

There are no open-loop zeros. $m = 0$.

Number of branches of root locus $N = n = 4$.

Number of asymptotes $= n - m = 4 - 0 = 4$.

Step-2 As the poles s_3 and s_4 are symmetrical about the real axis, the root locus will be symmetrical about real axis.

Step-3 The four branches of the root locus start at the open loop poles $s_1 = 0$, $s_2 = -2$, $s_3 = -3 + j4$, $s_4 = -3 - j4$ where $K = 0$. These branches terminate at the open loop zeros at infinity where $K = \infty$.

Step-4 The four branches of the root locus go to zeros at infinity along asymptotes. The angles of asymptotes are

$\theta_q = \frac{(2q+1) \times 180^\circ}{n-m}$, $q = 0, 1, 2, 3$.

$\because n - m - 1$
$q = 4 - 0 - 1 = 3$

Therefore,

for $q=0$, $\phi_{A1} = \frac{(2\times 0)+1}{4} \times 180^\circ = 45^\circ$

for $q=1$, $\phi_{A2} = \frac{(2\times 1)+1}{4} \times 180^\circ = 135^\circ$

for $q=2$, $\phi_{A3} = \frac{(2\times 2)+1}{4} \times 180^\circ = 225^\circ$

for $q=3$, $\phi_{A4} = \frac{(2\times 3)+1}{4} \times 180^\circ = 315^\circ$

Step-5 Centroid (σ_A) = $\frac{\text{(sum of real parts of poles} - \text{sum of real parts of zeros)}}{\text{number of poles} - \text{number of zeros}}$

$$= \frac{(0-2-3-3)-(0)}{4-0} = -2$$

Step-6 Breakaway point

K can be found from the solution of $\frac{dK}{ds} = 0$

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+2)(s^2+6s+25)} = 0$$

$$\Rightarrow \frac{s(s+2)(s^2+6s+25)+K}{s(s+2)(s^2+6s+25)} = 0$$

$$s(s+2)(s^2+6s+25)+K = 0$$

$$\Rightarrow K = -\{(s^2+2s)(s^2+6s+25)\}$$

$$= -\{s^4+6s^3+25s^2+2s^3+12s^2+50s\}$$

$$K = -\{s^4+8s^3+37s^2+50s\}$$

Scanned by CamScanner

$$\frac{dk}{ds} = \frac{d}{ds}\left\{-\left(s^4 + 8s^3 + 37s^2 + 50s\right)\right\} = 0$$

$$4s^3 + 24s^2 + 74s + 50 = 0$$

$$2s^3 + 12s^2 + 37s + 25 = 0$$

for this eq^n three roots are found

$s_1 = -0.9$

$s_2 = -2.55 + j2.72$

$s_3 = -2.55 + j2.72$.

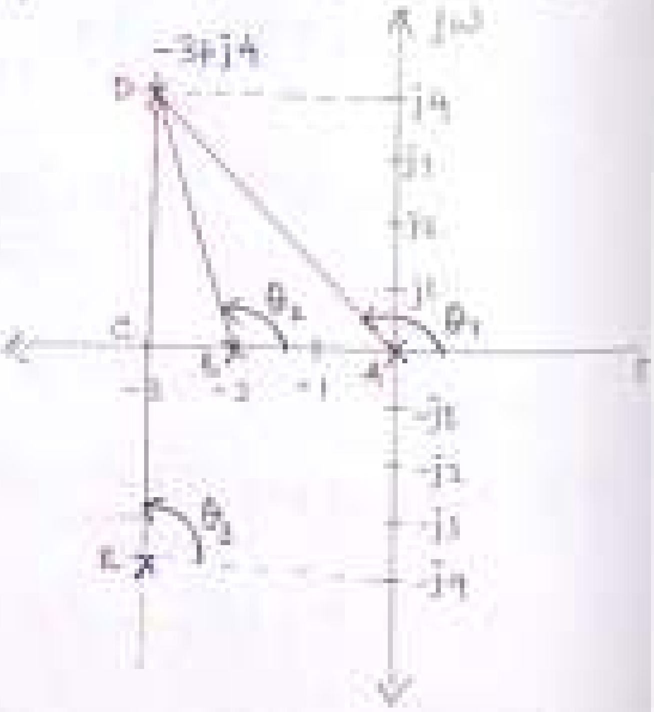
As these two points s_2, s_3 are not on the root locus, hence ignored. The only break away point is at $s = -0.9$

Step-7 The angle of departure from the complex pole $s = -3+j4$ is given by

$\phi_d = 180 - \phi$. $\left\{\text{we have two complex poles hence we need to calculate } \phi_{d1}, \phi_{d2}\right\}$

where $\phi = \theta_1 + \theta_2 + \theta_3$

The angles $\theta_1, \theta_2, \theta_3$ from the poles at $s = 0$, $s = -2$ and $s = -3-j4$ are found from the figure.



$\theta_1 = 180° - \tan^{-1}\frac{DC}{CA}$

$= 180° - \tan^{-1}\frac{4}{3} = 180 - 53° = 127°$

$\theta_2 = 180° - \tan^{-1}\frac{DC}{CB} = 180° - \tan^{-1}\frac{4}{1}$

$= 180° - 75.96°$

$= 104°$

$\theta_3 = 90°$

Therefore $\theta = \theta_1 + \theta_2 + \theta_3$

$$= 127^\circ + 104^\circ + 90^\circ$$

$$= 321^\circ$$

$$\phi_{A_1} = 180^\circ - \theta = 180^\circ - 321^\circ = -141^\circ$$

$$\phi_{A_2} = -\phi_{A_1} = -(-141^\circ) = 141^\circ$$

Step-5 Point of intersection of the root locus with the imaginary axis

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+2)(s^2+6s+25)} = 0$$

$$s^4 + 8s^3 + 37s^2 + 50s + K = 0$$

Putting $s = j\omega$ in the above eqn

$$(j\omega)^4 + 8(j\omega)^3 + 37(j\omega)^2 + 50(j\omega) + K = 0$$

$$\omega^4 + (-j8\omega^3) + (-37\omega^2) + 50j\omega + K = 0$$

Separating real part and imaginary parts

$$(\omega^4 - 37\omega^2 + K) + j(-8\omega^3 + 50\omega) = 0$$

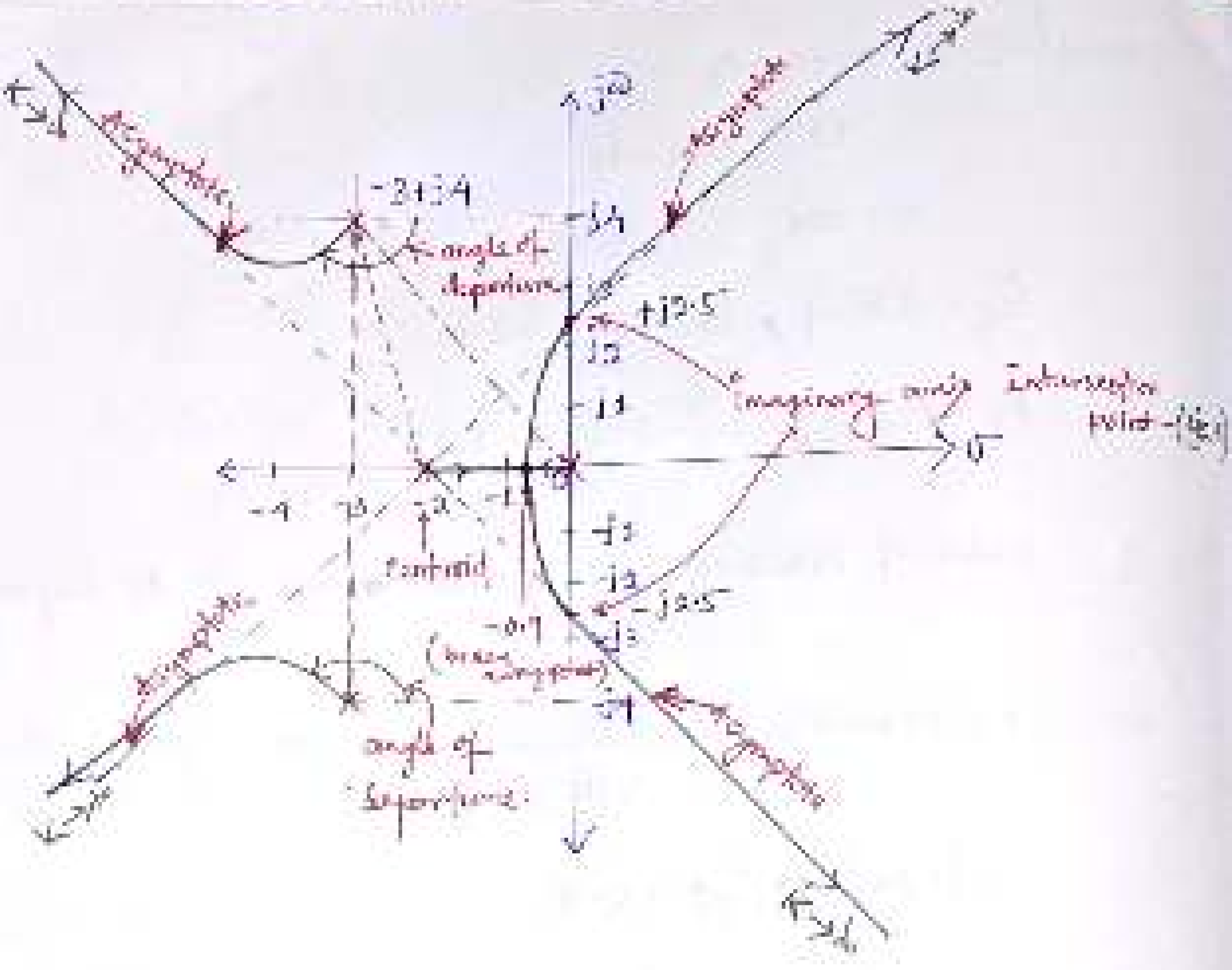
Equating imaginary part to zero

$$-8\omega^3 + 50\omega = 0$$

$$-8\omega^2 + 50 = 0$$

$$8\omega^2 = 50$$

$$\omega = \pm\sqrt{50/8} = \pm 2.5 \text{ rad/sec}$$



Scanned by CamScanner

Methods to improve time response

It is necessary for a control system to meet certain specifications reflecting the performance. System performance can be improved by using any of the following control methods.

i) Proportional control

ii) Proportional plus Derivative control

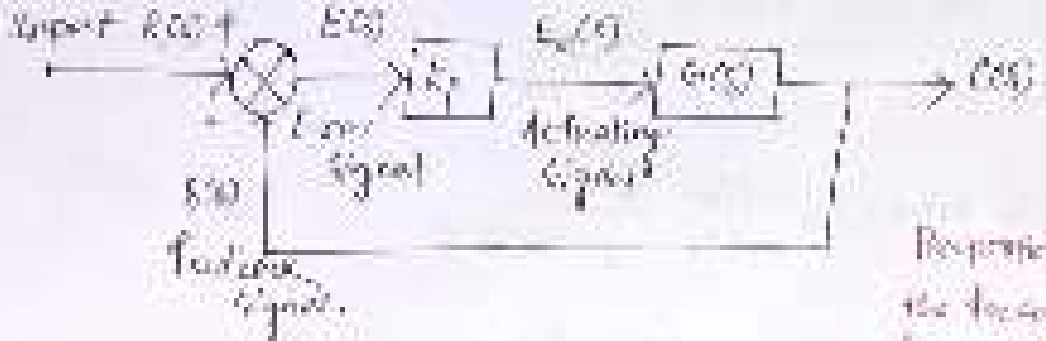
iii) Proportional plus Integral control

iv) Proportional plus Integral plus Derivative control.

A controller is a device which is introduced in feedback or forward path of a system, to control the steady-state and transient than according to the requirements.

Proportional Controller (P controller)

The proportional controller is a device that produces a control signal which is proportional to the input error signal.



Proportional controller amplifies the forward path gain. If forward path gain is increased the peak overshoot increases but steady state error is reduced.

From this figure, $e_a(t) = K_p e(t)$

$$E_a(s) = K_p E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p}$$

Proportional plus Derivative Controller (PD controller)

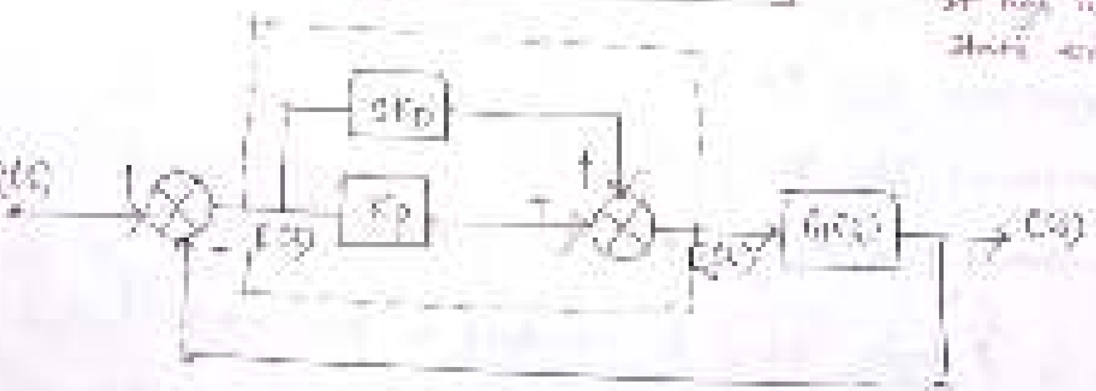
In this controller, the activating signal $E_a(s)$ is proportional to the error signal $e(t)$ and with proportional to derivative of the error signal. Thus the activating signal will be

$$e_a(t) = K_p e(t) + K_D \frac{d}{dt} e(t)$$

$$E_a(s) = K_p E(s) + K_D s E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p + K_D s}$$

By using this PD controller overshoot can be reduced but derivative alone is ... it has no effect on steady state error.



Proportional plus Integral Controller (PI controller)

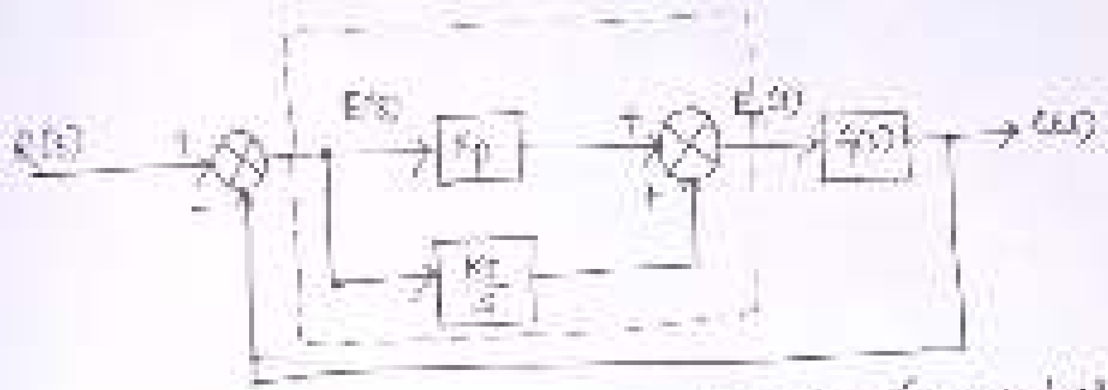
In this controller the actuating signal consists of proportional error signal adding to the integral of error signal.

$$E_a(t) = K_p e(t) + K_I \int e(t)\, dt$$

$$E_a(s) = K_p E(s) + \frac{K_I}{s} E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p + \frac{K_I}{s}}$$

By using this type controller we can improve the order of the system by one, which results in the reduction of the steady state error. But the system stability may be hampered.



Proportional plus Integral plus Derivative Controller (PID controller)

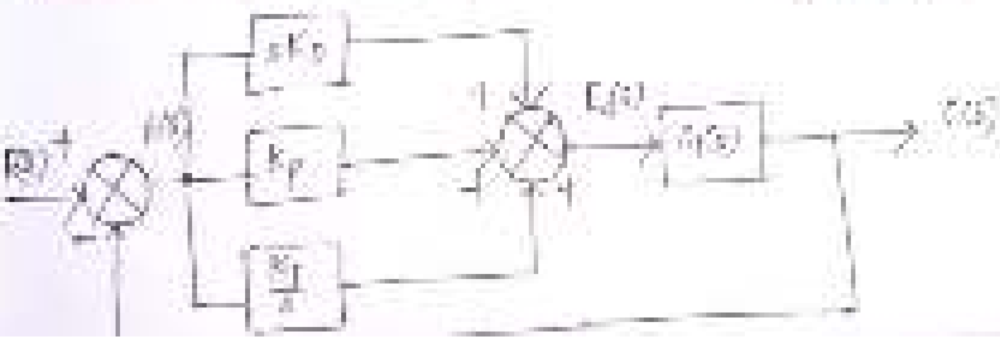
In this controller the actuating signal is proportional to error signal, to integral of error signal, to derivative of error signal.

$$E_a(t) = K_p e(t) + K_I \int e(t)\, dt + K_D \frac{d}{dt} e(t)$$

$$E_a(s) = K_p E(s) + \frac{K_I}{s} E(s) + sK_D E(s)$$

$$\boxed{\frac{E_a(s)}{E(s)} = K_p + sK_D + \frac{K_I}{s}}$$

The proportional controller stability though does not produces steady state error. The integral controller reduces the steady state error. The derivative controller decreases the rate of change of error. The main advantages of PID controller are higher stability, no offset and reduced over shoot.



Effect of adding poles to G(s)H(s)

1. There is a change in shape of the root locus and shifts towards the right half of the s-plane.
2. The angle of asymptotes reduce.
3. The centroid is shifted to the left.
4. The relative stability of the system is decreased.
5. There is a reduction in the range of K. Even a system which was perfectly stable may become unstable as K increases.

Effect of adding zeros to G(s)H(s)

1. There is a change in the shape of the root locus and it shifts towards the left half of the s plane.
2. The relative stability of the system is improved.

Frequency Response Analysis

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.

The response of a system to a sinusoidal input signal is an output sinusoidal signal at the same frequency as the input. However, the magnitude and phase of the output signal differ from those of the input sinusoidal signal.

As the frequency varies from 0 to ∞ the magnitude and phase angle vary. Variation in magnitude (M) and phase (ϕ) of $G(j\omega)$ as the input frequency is varied from 0 to ∞ can be obtained by different methods.

(i) Polar plot (ii) Bode plot iii) Nyquist plot

Frequency domain specifications :-

Frequency domain specifications are the measures of the performance and characteristics of a control system.

i) Resonant Peak Magnitude (M_r)
ii) Resonant frequency (ω_r)

M_r – The resonant peak magnitude is the maximum value of the closed-loop frequency response.

ω_r – The frequency at which the system has maximum magnitude is called resonant frequency.

Scanned by CamScanner

Correlation between time and frequency response:

For a peak overshoot M_p in time domain, there exists a corresponding resonant peak M_r, in the frequency domain.

For a damped frequency ω_d in the time domain, there is a corresponding resonant frequency ω_r in the frequency domain.

All these expressions can be expressed in terms of ξ and ω_n.

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

Example – The specification on a second-order unity feedback control system with the closed-loop transfer function, are that the

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

overshoot of ~~the~~ step response should not exceed 12% and the peak time must be less than 0.2s. Find the corresponding frequency-response value of peak resonant magnitude and resonant frequency.

Solution — Percent maximum Overshoot.

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 12\% = \frac{12}{100}$$

Taking natural logarithm on both side

$$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = \ln\frac{12}{100} = -2.12$$

Scanned by CamScanner

Squaring both sides,

$$\frac{\pi^2 \xi^2}{\left(\sqrt{1-\xi^2}\right)^2} = 4.495$$

$$\xi^2 = \frac{4.495}{\pi^2 + 4.495} = 0.3129$$

$$\xi = \sqrt{0.3129} = 0.5594$$

Resonant Peak, $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

$$= \frac{1}{2\times 0.5594\sqrt{1-(0.5594)^2}} = 1.078$$

Peak time, $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$$0.2 = \frac{\pi}{\omega_n\sqrt{1-(0.5594)^2}}$$

$$\omega_n = \frac{\pi}{0.2\sqrt{1-(0.5594)^2}} = 18.95 \text{ rad/sec}$$

Resonant frequency, $\omega_r = \omega_n\sqrt{1-2\xi^2}$

$$= 18.95\sqrt{1-2(0.5594)^2}$$

$$= 11.59 \text{ rad/sec}$$

Scanned by CamScanner

Polar Plots

The polar plot of a function of a complex variable i.e. G(s) The polar plot of a function is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity.

Hence polar plot is the locus of phasor $|G(j\omega)| \angle G(j\omega)$ as 'ω' is varied from zero to infinity.

In a casual way, the polar plot is often called the 'Nyquist plot'. The polar plot is strictly for the frequency range $0 < \omega < \infty$, while the Nyquist plot is in the frequency range $-\infty < \omega < \infty$.

Procedure to sketch the polar plot :-

If we know the value of magnitude (M) and phase (Φ) at ω=0 and ω=∞, then we can sketch the polar plot.

Steps to be followed:-

1) Determine the gain of the system (G(s))
2) Put s = jω in the gain to obtain G(jω).
3) Calculate the magnitude of G(jω)

$$|G(j\omega)|_{\omega=0} = \lim_{\omega \to 0} |G(j\omega)|$$

$$|G(j\omega)|_{\omega=\infty} = \lim_{\omega \to \infty} |G(j\omega)|$$

4) Calculate the phase angle of G(jω).

$$\angle G(j\omega)\big|_{\omega=0} = \lim_{\omega \to 0} \angle G(j\omega)$$

$$\angle G(j\omega)\big|_{\omega=\infty} = \lim_{\omega \to \infty} \angle G(j\omega)$$

5) To find the frequencies at which the polar plot crosses the real and imaginary axis, we rationalize the complex frequency function $G(j\omega)$, we multiply numerator and denominator of $G(j\omega)$ with complex conjugate of the denominator factors. Separate the real and imaginary parts.

6) Equate the imaginary part of $G(j\omega)$ to zero to determine the values of the frequency ω at which the polar plot intersects the real axis. Calculate the values of $G(j\omega)$ at the intersecting points by substituting the determined values of frequency ω in the rationalized expression of $G(j\omega)$.

7) Equate the real part of $G(j\omega)$ to zero and determine the value of the frequency ω at which the plot intersects the imaginary axis. Calculate the value of $G(j\omega)$ at the points of intersection by substituting the determined values of frequency ω in the rationalized expression of $G(j\omega)$.

8) Sketch the polar plot from above information.

Example - Sketch the polar plot of $G(s) = 1 + sT$

Solution. Putting $s = j\omega$

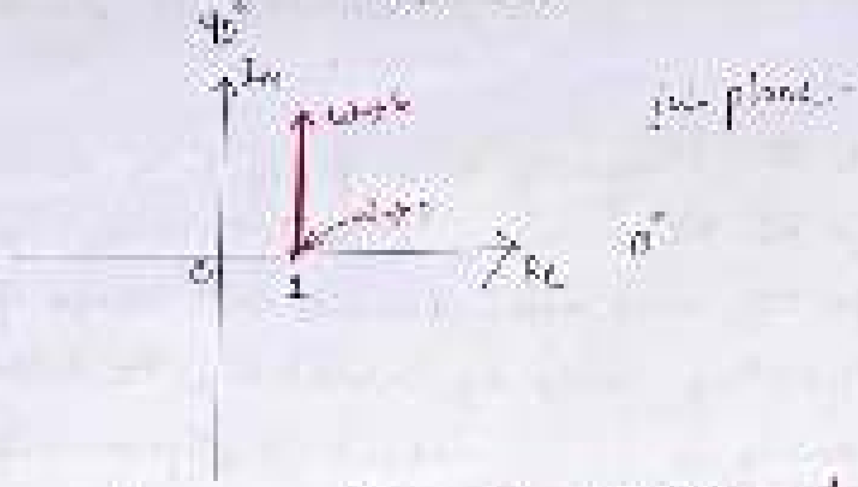
$$G(j\omega) = 1 + j\omega T$$

$$M = |G(j\omega)| = \sqrt{1^2 + (\omega T)^2}$$

$$\phi = \angle G(j\omega) = \tan^{-1} \omega T$$

$$M\Big|_{\omega\to 0} = \lim_{\omega\to 0} \sqrt{1+\omega^2T^2} = 1 \quad , \quad \phi\Big|_{\omega\to 0} = \lim_{\omega\to 0} \tan^{-1}\omega T = 0^\circ$$

$$M\Big|_{\omega\to\infty} = \lim_{\omega\to\infty} \sqrt{1+\omega^2T^2} = \infty \quad , \quad \phi\Big|_{\omega\to\infty} = \lim_{\omega\to\infty} \tan^{-1}\omega T = 90^\circ$$



Example:- Sketch the polar plot of the function $G(s) = \dfrac{1}{s(1+sT)}$

Solution:- The sinusoidal transfer function is obtained by putting $s=j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$

$$G(j\omega) = \frac{1\angle 0}{(0+j\omega)(1+j\omega T)}$$

The transfer function can be written as

$$G(j\omega) = |G(j\omega)|\angle G(j\omega) = M\angle\phi$$

$$M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2T^2}}$$

$$\phi = \angle G(j\omega) = \frac{\tan^{-1}\left(\frac{0}{1}\right)}{\tan^{-1}\left(\frac{\omega}{0}\right)\ \tan^{-1}\left(\frac{\omega T}{1}\right)}$$

$$= \frac{0^\circ}{90^\circ + \tan^{-1}(\omega T)}$$

$\phi = -90^\circ - \tan^{-1}\omega T$

$$M\Big|_{\omega=0} = \lim_{\omega\to 0} |G(j\omega)| = \lim_{\omega\to 0} \frac{1}{0} = \infty$$

$$\phi\Big|_{\omega=0} = \lim_{\omega\to 0} \angle G(j\omega) = -90^\circ - \tan^{-1}0 = -90^\circ$$

$$G(j\omega)\Big|_{\omega=0} = \infty \angle -90^\circ$$

$$M\Big|_{\omega=\infty} = \lim_{\omega\to\infty} |G(j\omega)| = \frac{1}{\infty} = 0$$

$$\phi\Big|_{\omega=\infty} = \lim_{\omega\to\infty} \angle G(j\omega) = -90^\circ - 90^\circ = -180^\circ$$

$$G(j\omega)\Big|_{\omega=\infty} = 0 \angle -180^\circ$$

From above data we can't say whether the function approaches infinity asymptotically to the -90° axis or some line parallel to it. The true asymptote is determined by finding the value of real part of $G(j\omega)$ as ω approaches zero.

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$

Rationalizing, we get $G(j\omega) = \dfrac{-j\omega(1-j\omega T)}{j\omega(1+j\omega T)(-j\omega)(1-j\omega T)}$

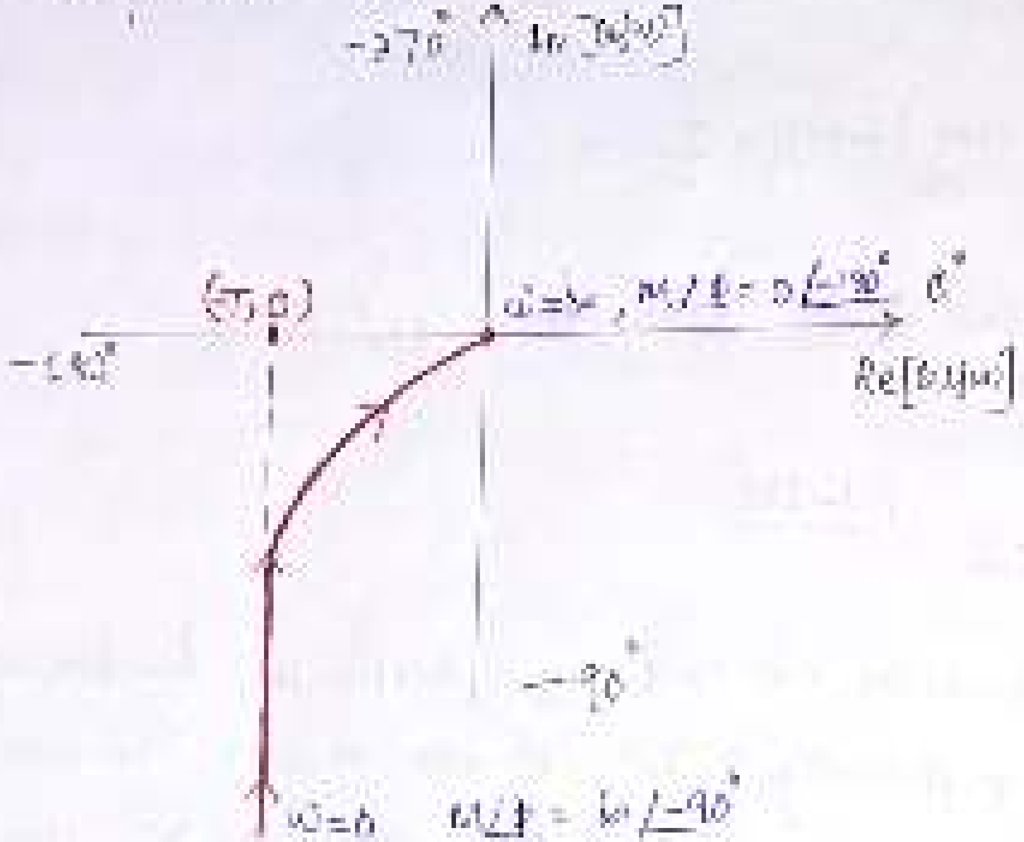
$$= \frac{-j\omega - \omega^2 T}{\omega^2(1+\omega^2 T^2)}$$

or $$G(j\omega) = \frac{-T}{1+\omega^2T^2} - j\frac{1}{\omega(1+\omega^2T^2)}$$

$$\lim_{\omega \to \infty} Re[G(j\omega)]$$

$$\lim_{\omega \to 0} \left[\frac{-T}{1+\omega^2T^2}\right] = -T.$$

Hence, the plot is asymptotic to the vertical line passing through the point $(-T, 0)$



Bode Plot

A bode plot is a graphical method to determine the frequency response of a system. Bode plot consists of two separate plots. One shows how the amplitude of $G(j\omega)$ varies with frequency, and the other shows how the phase angle of $G(j\omega)$ varies with frequency. Bode plots are made in semilog graph paper. In both the amplitude and phase plot, frequency is plotted in horizontal log scale and amplitude and phase angle are plotted in the vertical scale.

A graph paper with horizontal axis in logarithmic scale and vertical axis in linear scale is called semilog paper.

Bode magnitude plot

It is the plot of the logarithm of the magnitude of a sinusoidal transfer function $G(j\omega)$ versus $\log \omega$ where ω is the radian frequency. The standard representation of logarithmic magnitude is $20 \log|G(j\omega)|$. The unit is decibel (dB).

Bode Phase angle plot

It is the plot of the phase angle in degrees of the sinusoidal transfer function versus $\log_{10} \omega$.

Ⓝ It is to be noted that the point $\omega = 0$ on the log scale cannot be located as $\log 0 = -\infty$.

Standard form of $G(j\omega)$

Consider the open loop transfer function in the time constant form

$$G(s) = \frac{K(1+sT_1)(1+sT_2)}{s(1+sT_3)(1+sT_4)}$$

The sinusoidal transfer function is obtained by putting $s = j\omega$

$$G(j\omega) = \frac{K(1+j\omega T_1)(1+j\omega T_2)}{j\omega(1+j\omega T_3)(1+j\omega T_4)}$$

This equation can be written in polar form as

$$G(j\omega) = \frac{K\angle 0^\circ\left[\sqrt{1+\omega^2T_1^2}\,(\tan^{-1}\omega T_1)\right]\left[\sqrt{1+\omega^2T_2^2}\,(\tan^{-1}\omega T_2)\right]}{\omega\angle 90^\circ\left[\sqrt{1+\omega^2T_3^2}\,(\tan^{-1}\omega T_3)\right]\left[\sqrt{1+\omega^2T_4^2}\,(\tan^{-1}\omega T_4)\right]}$$

$$|G(j\omega)| = \frac{K\sqrt{1+\omega^2T_1^2}\,\sqrt{1+\omega^2T_2^2}}{\omega\sqrt{1+\omega^2T_3^2}\,\sqrt{1+\omega^2T_4^2}}$$

$$\angle G(j\omega) = 0^\circ + \tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 - 90^\circ - \tan^{-1}\omega T_3 - \tan^{-1}\omega T_4.$$

The logarithmic magnitude of $G(j\omega)$ in decibels (dB) is obtained by multiplying the logarithm to the base 10 of $G(j\omega)$ by 20.

So, log magnitude can be written as $20 \times \log_{10}|G(j\omega)|$ dB.

Scanned by CamScanner

$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{K\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}{\omega\sqrt{1+\omega^2T_3^2}\sqrt{1+\omega^2T_4^2}}$$

$$= 20\log K + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\sqrt{1+\omega^2T_2^2} - 20\log\omega - 20\log\sqrt{1+\omega^2T_3^2} - 20\log\sqrt{1+\omega^2T_4^2}$$

Bode plot of Constant Gain 'K'

Let $G(s) = K$.

$G(j\omega) = K$.

$|G(j\omega)| = K$.

Magnitude plot $\Rightarrow$ $20\log|G(j\omega)| = 20\log K$.

let magnitude in dB $= 20\log|G(j\omega)| = A$ dB.

here $A = 20\log K$.

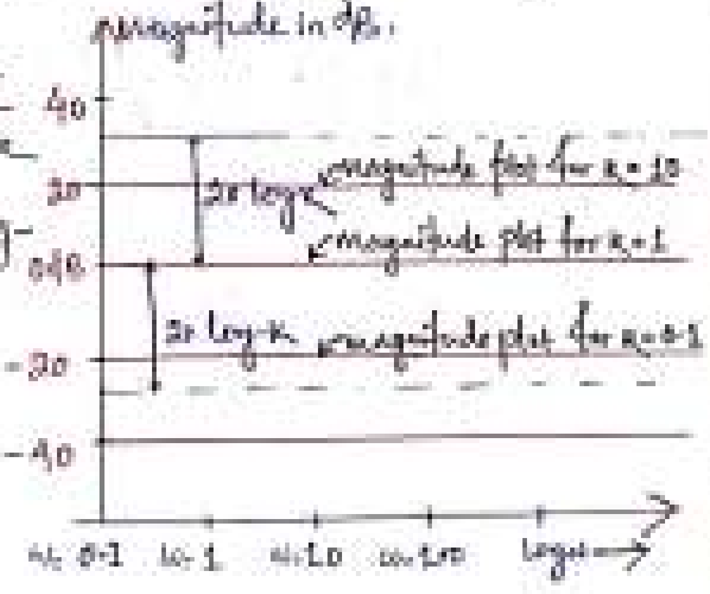
when $K>1$, $A = 20\log K$ = a positive Constant.

when $K=1$, $A = 20\log 1 = 0$.

when $K<1$, $A = 20\log K$ = a negative Constant.

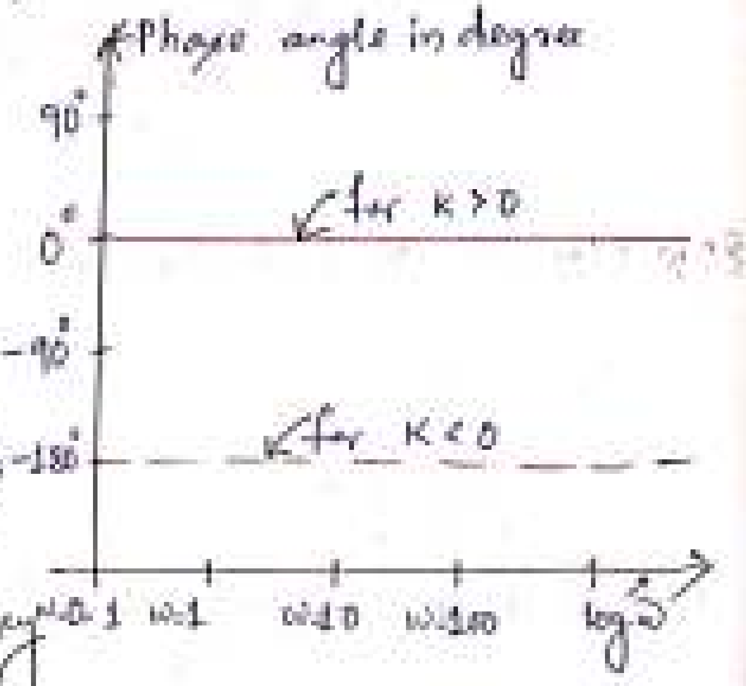
Here we can see the magnitude 'A' is independent of 'log ω'

If we draw the magnitude plot of above function then it will be a graph between 'log ω' and magnitude in dB.



$G(j\omega) = K + j0 = K\angle 0^\circ$

Thus phase angle is $\phi = 0^\circ$.
What ever the value of 'ω', phase angle will be zero.
The effect of varying 'K' has no effect on the phase angle plot.
If the gain K is negative, then negative sign attributes -180° to the phase angle plot which is independent of frequency 'ω'.



So phase angle plot is the graph between phase angle in degree and log of frequency log ω.

Bode plot of Integral factor :—

Magnitude plot

(a) Let $G(s) = 1/s$ (Single pole at origin)

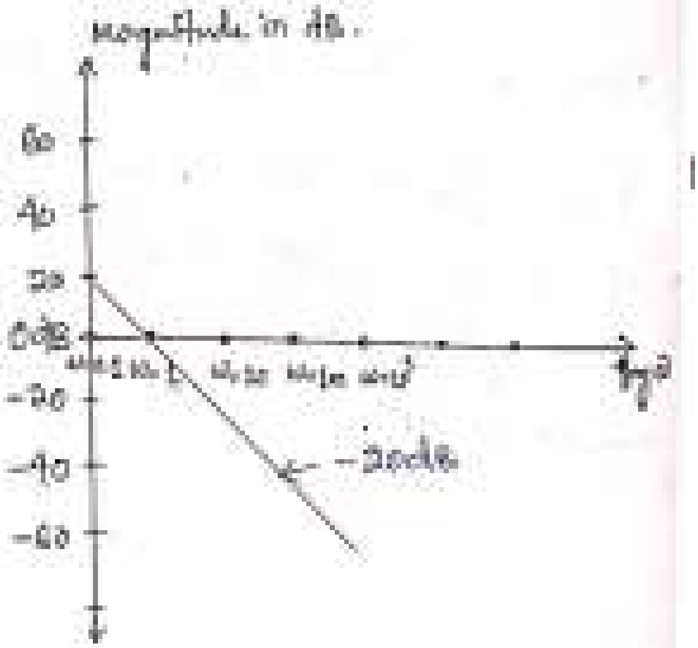
$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{0 + j\omega}$$

$$|G(j\omega)| = \frac{1}{\sqrt{0^2 + \omega^2}} = \frac{1}{\omega}$$

log magnitude: $20\log|G(j\omega)| = 20\log\frac{1}{\omega}$

$$= 20\log\omega^{-1}$$

$$= -20\log\omega.$$



Scanned by CamScanner

At $\omega = 1$, $-20 \log 1 = 0$ dB.

At $\omega = 10$, $-20 \log 10 = -20$ dB.

At $\omega = 100$, $-20 \log 100 = -40$ dB.

At $\omega = 1000$, $-20 \log 1000 = -60$ dB.

(* It is seen that as the frequency increases by a factor of 10, the corresponding decibel value increases by a factor 20.

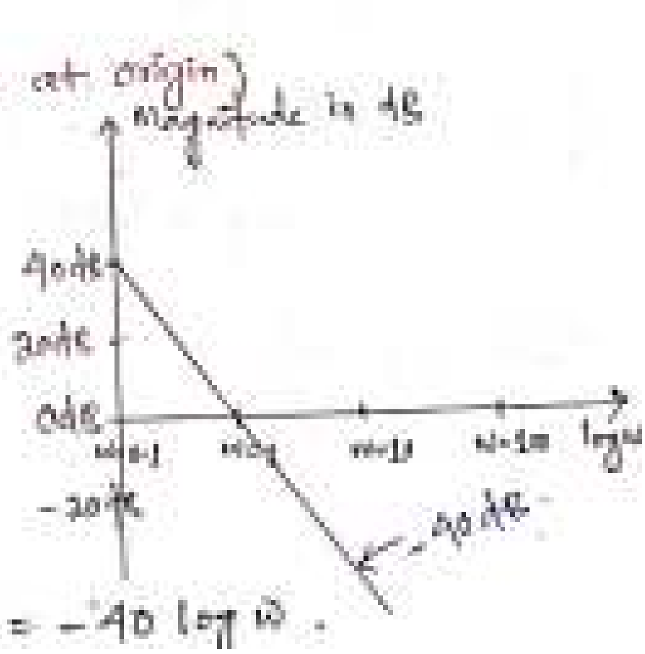
(* In a bode plot frequency ratios are expressed in terms of 'DECADES' → An interval of two frequencies with a ratio equal to 10 is called decades.

(* Sometime in place of decade, 'OCTAVES' are also used. An interval of two frequencies with a ratio equal to 2 is called octaves.

(* The relation between these two expression —

$$\boxed{20\,\text{dB}/\text{decade} = 6\,\text{dB}/\text{octave}}$$

(b) Let $G(s) = \frac{1}{s^2}$ (Two poles at origin)



$G(j\omega) = \frac{1}{(j\omega)^2}$

$$|G(j\omega)| \text{ in dB} = 20 \log \left| \frac{1}{(j\omega)^2} \right|$$

$$= 20 \log \frac{1}{\omega^2}$$

$$= 20 \log \omega^{-2} = -40 \log \omega.$$

This represents the equation of a straight line having slope of -40 dB/decade and it passes through the 0 dB point at $\omega=1$

(c) Let $G(s) = \frac{1}{s^n}$ (multiple poles at origin)

$$G(j\omega) = \frac{1}{(j\omega)^n}$$

log magnitude of $G(j\omega)$ in dB $= 20\log\left|\frac{1}{(j\omega)^n}\right|$

$$= 20\log\omega^{-n}$$

$$= -n\,20\log\omega$$

$$= -20n\log\omega.$$

This represents a straight line having slope of $-20n$ dB/decade and it passes through the 0 dB point in the logarithmic frequency scale on semilog paper.

Bode plot of Derivative factor :-

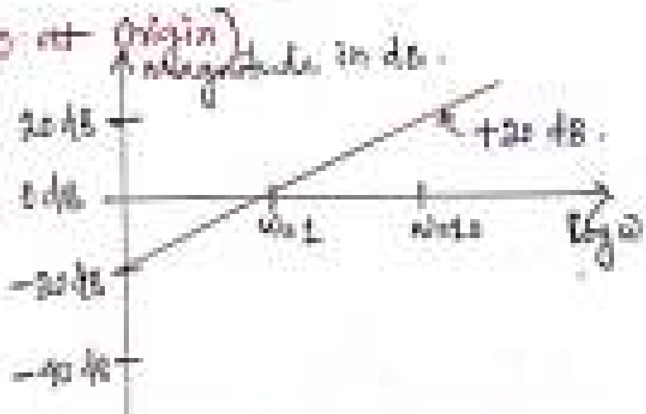
Magnitude plot

(a) Let $G(s) = s$ (Single zero at origin)

$$G(j\omega) = j\omega$$

$$|G(j\omega)| \text{ in dB} = 20\log|j\omega|$$

$$= +20\log\omega$$



This represents the equation of a straight line whose slope is $+20$ dB/decade and it passes the 0 dB point at $\omega=1$.

(b) Let $G(s) = s^n$ (multiple zeros at origin).

$G(j\omega) = (j\omega)^n$

$|G(j\omega)|$ in dB $= 20\log|(j\omega)^n|$

$= +20n\log\omega$.

This represents straight line having slope of $+20n$ dB/decade which passes through the 0dB point at $\omega = 1$ in the log frequency scale.

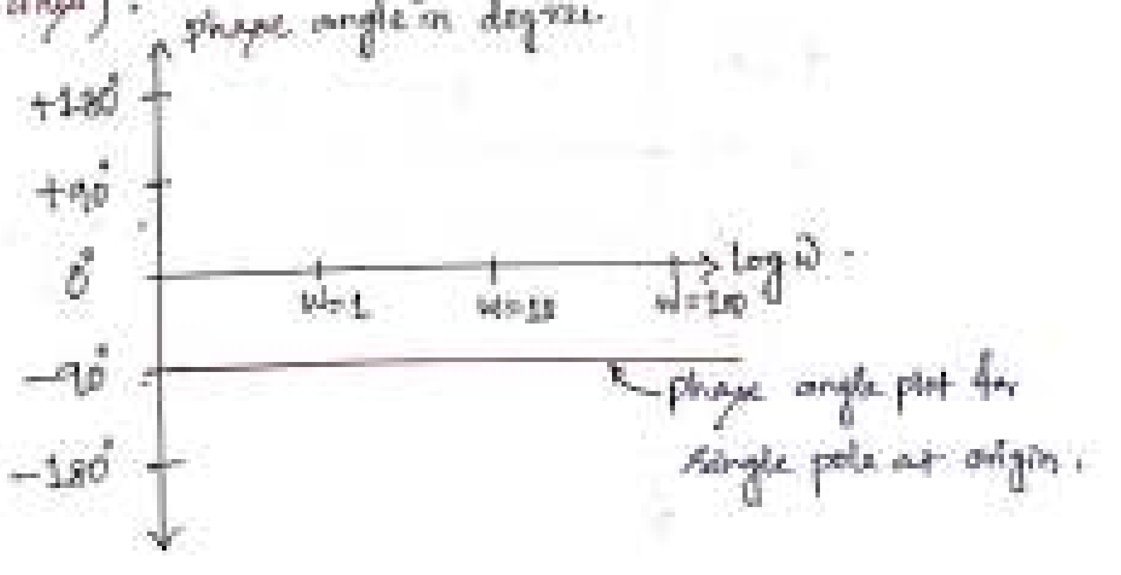
Phase angle plot for integral factor

(a) Let $G(s) = \frac{1}{s}$ (one pole at origin)

$G(j\omega) = \frac{1}{j\omega}$

$\angle G(j\omega) = \angle\frac{1}{j\omega} = \frac{\angle 0^\circ}{\angle 90^\circ} = -90^\circ$.

It is seen that phase angle is independent of ω. Hence the phase angle plot of one pole at origin of the s-plane is a line parallel to the horizontal axis (log ω axis).



(b) Let $G(s) = \frac{1}{s^n}$ (n poles at origin)

$$G(j\omega) = \frac{1}{(j\omega)^n} = \frac{1}{j\omega} \cdot \frac{1}{j\omega} \cdot \frac{1}{j\omega} \ldots\ldots \text{n times.}$$

$$\angle G(j\omega) = \frac{\angle 0^\circ}{90^\circ} \cdot \frac{\angle 0^\circ}{90^\circ} \cdot \frac{\angle 0^\circ}{90^\circ} \ldots\ldots \text{n times..}$$

$$\angle G(j\omega) = n \times \frac{\angle 0^\circ}{90^\circ} = -n 90^\circ.$$

Thus, n poles at the origin of s plane contribute $(-n \times 90^\circ)$ angle to overall phase angle plot.

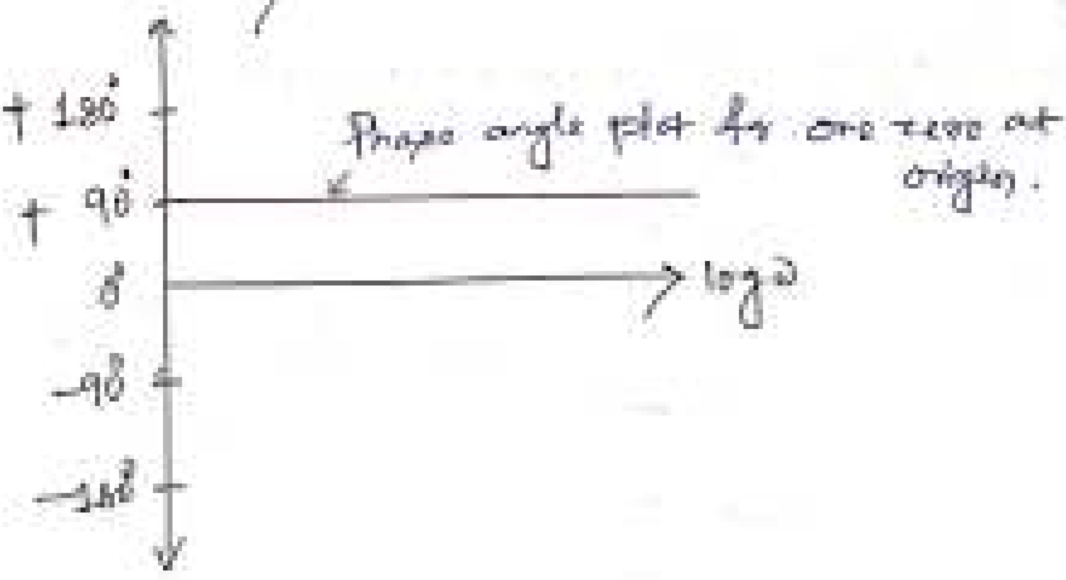
$\boxed{\text{Phase angle plot for derivative factor}}$

(a) Let $G(s) = s$ (one zero at origin):

$G(j\omega) = j\omega$

$\angle G(j\omega) = \angle j\omega = +90^\circ$

Thus for a single zero at the origin of s-plane has a constant value $+90^\circ$ for all frequencies. This line will be parallel to horizontal axis.



(ii) Let $G(s) = s^{n}$ (n zeros at origin)

$$G(j\omega) = (j\omega)^{n} = (j\omega)(j\omega)(j\omega) \text{ ...... n times.}$$

$$\angle G(j\omega) = +90^{\circ} + 90^{\circ} + 90^{\circ} \text{ ...... n times.}$$

$$= +n \times 90^{\circ}.$$

Hence 'n' zeros at origin contribute $+n \times 90^{\circ}$ angle to the overall phase angle plot.

Bode plot of first-order factors in the denominator

Let $G(s) = \dfrac{1}{1+sT}$

The sinusoidal transfer function will be

$$G(j\omega) = \frac{1}{1+j\omega T}$$

In polar form $G(j\omega) = |G(j\omega)| \angle G(j\omega) = M\angle\phi$

Bode magnitude plot:

Log-magnitude in dB = $20 \log M$

$$\Rightarrow A = 20 \log \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$\Rightarrow A = 20 \log \left(\sqrt{1+\omega^2T^2}\right)^{-1}$$

$$\Rightarrow A = -20 \log \left(\sqrt{1+\omega^2T^2}\right) \text{ dB.}$$

Low frequencies :—

When $T << 1$, T is negligible compared to 1. Therefore,

$$A = -20 \log \sqrt{1+0}$$
$$= -20 \log 1$$
$$= 0 dB.$$

Hence log magnitude curve at low frequencies is the constant 0 dB line.

High frequencies :—

if $\omega >> \frac{1}{T}$, 1 can be neglected in comparison with $\omega^2 T^2$. Then

$$20 \log |G(j\omega)| = A = -20 \log \sqrt{1+\omega^2 T^2}$$
$$A = -20 \log \omega T \text{ dB}.$$

at $\omega = 1/T$, the log magnitude is equal to 0 dB.

at $\omega = \frac{10}{T}$, the " " " " " 20 dB.

Hence, the value of $-20 \log \omega T$ dB decreases by 20 dB for each decade of ω.

Hence there is two lines, one for lower frequency range $1 < \omega < 1/T$ and the other for $1/T < \omega < \infty$, the high frequency range.

The intersection point for these two lines can be found by equating $-20 \log \omega T = 0$

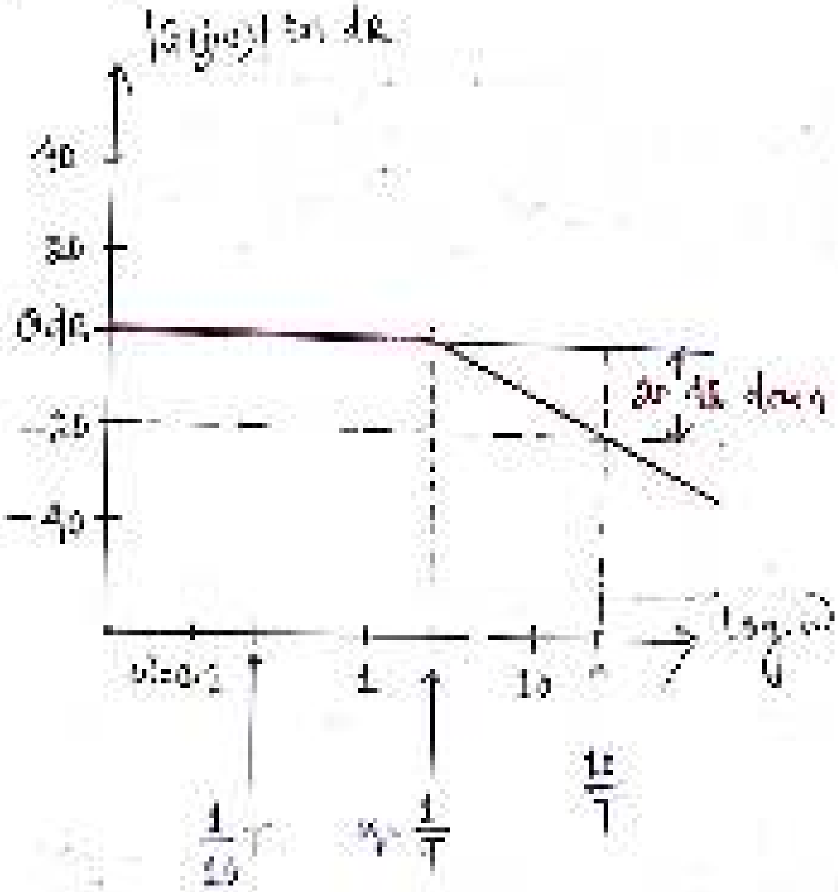
$$\Rightarrow \omega T = 1 \Rightarrow \omega = \frac{1}{T}$$

These two straight lines are called 'straight-line asymptotes'. The frequency at which the asymptotes meet is called the 'corner frequency'. It is denoted as 'ω_c'.

Hence, the corner frequency ω_c divides the frequency response curve in to two regions, low frequency region and high frequency region.

A. 0 dB line up to $\omega_c = 1/T$

B. line of slope -20 dB/decade when $\omega > \omega_c$



Phase Angle Plot :-

$$G(j\omega) = \frac{1}{1+j\omega T}$$

The phase angle $\angle G(j\omega) = -\tan^{-1}\frac{\omega T}{1} = -\tan^{-1}\omega T = \phi$

When $\omega = 0$, $\phi = 0^\circ$

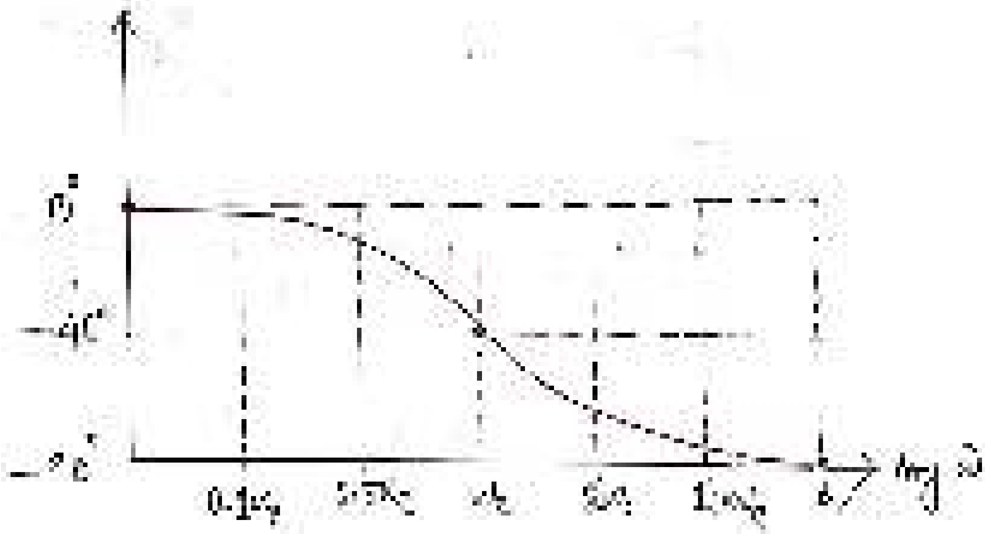
When $\omega = \omega_c = \frac{1}{T}$

$$\phi = -\tan^{-1}\left(\frac{1}{T}\cdot T\right) = \tan^{-1}(1) = -45^\circ$$

At very high frequency ωT is very large so

$$\phi = -\tan^{-1}(\infty) = -90^\circ$$

ω	$0.1\omega_c$	$0.5\omega_c$	ω_c	$2\omega_c$	$10\omega_c$	∞
$\angle G(j\omega)$	-5.7°	-26.6°	-45°	-63.4°	-84.3°	-90°



Scanned by CamScanner

Relative Stability

Two commonly used measures for relative stability are

i) Gain margin ii) phase margin.

Before going deep in to these two, we should know about another two terms a) Gain Cross Over frequency (ω_{gc})

b) Phase Cross over frequency (ω_{pc})

Gain Cross Over frequency

- The frequency at which the magnitude of $G(j\omega)H(j\omega)$ is unity is called gain cross over frequency.

Hence at gain cross over frequency

$$|G(j\omega)H(j\omega)| = 1.$$

In decibel if we want $20\log|G(j\omega)H(j\omega)| = 20\log 1 = 0\,dB$.

- Gain cross over frequency is the frequency at which the magnitude of $G(j\omega)H(j\omega)$ is 0 dB.

Phase Cross Over frequency

- The frequency at which phase angle of $G(j\omega)H(j\omega)$ is 180° is called phase cross over frequency.

At phase cross over frequency $\angle G(j\omega)H(j\omega) = 180^\circ$.

Gain Margin (GM)

GM is defined as the additional gain required to make the system just unstable.

Mathematically —

gain margin is the reciprocal of the magnitude of $G(j\omega)H(j\omega)$ measured at phase crossover frequency.

$$GM = \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$$

Gain Margin in decibels

$$GM = 20 \log \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$$

$$\boxed{GM = -20 \log |G(j\omega)H(j\omega)|_{\omega=\omega_{pc}} \text{ dB}}$$

Phase Margin (PM)

The phase margin is defined as the amount of additional phase lag which can be introduced in the system till the system becomes just unstable. It is measured at gain cross over frequency.

$$PM = 180^\circ + \phi$$

where $\phi = \angle G(j\omega_{gc})H(j\omega_{gc})$

Note :—

(vi) A large gain margin or a large phase margin indicate a very stable feed back system but usually a very sluggish.

(v) These measures of stability are valid for open loop stable systems only.

Determination of GM and PM from Bode plot

(i) The point at which the magnitude plot crosses 0 dB line is the gain cross over frequency, ω_{gc}.

(ii) The point at which the phase plot crosses the -180° line is the phase cross over frequency.

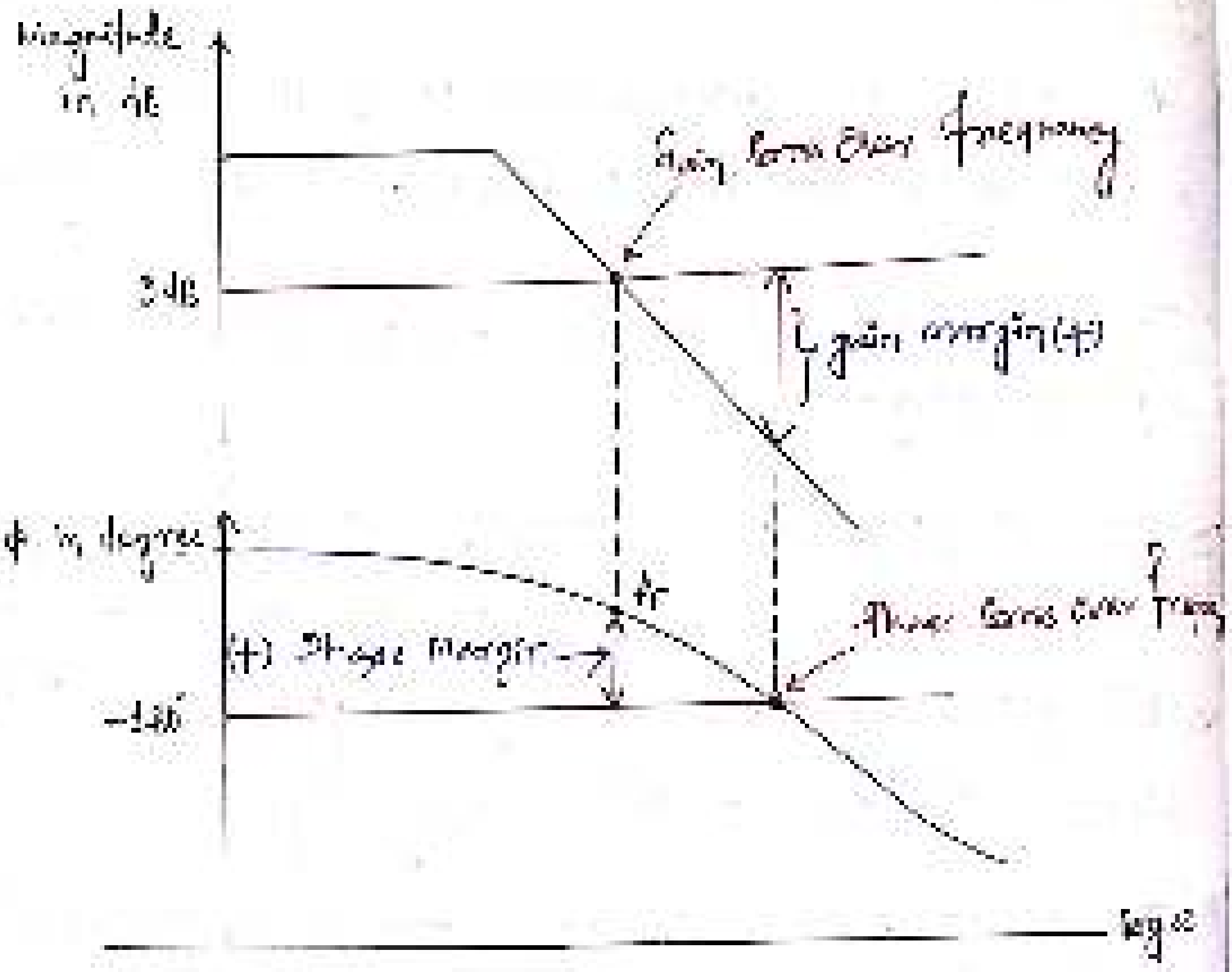
Phase margin —

(i) First we find the point at which the magnitude plot intersects the 0 dB line. This point is gain cross over frequency point (ω_{gc}).

(ii) Draw a vertical line from the gain crossover point till intersects the phase angle plot. Let this point be 'A'.

(iii) The difference between -180° line and 'A' point is the phase margin.

(iv) If the point 'A' on the phase angle plot is above -180° line, the phase margin is positive and if the point 'A' is below -180° line then phase margin is negative.



Gain Margin —

(i) First find the point at which the phase plot intersects the -180° line. This point is phase cross over frequency.

(ii) Draw a vertical line from the phase cross over point and it intersects the magnitude plot. Let the point be 'B'.

(iii) The distance between point B and 0 dB line is gain margin.

(iv) If point B is below 0 dB line, the gain margin is positive and if point B is above 0 dB line, the gain margin is negative.

Ques:— Construct the Bode plot for a unity feedback system

$$G(s) = \frac{[illegible]}{s(s+2)(s+50)}$$

From the Bode plot, determine,

a) Gain cross over frequency
b) Phase cross over frequency
c) Gain margin
d) Phase margin

Solution. $$G(s) = \frac{2000}{s(s+1)(s+100)}$$

Step 1 The given transfer function is in pole-zero form. Therefore, convert it into time-constant form.

$$G(s) = \frac{2000}{s(s+1)100\left(1+\frac{s}{100}\right)} = \frac{20}{s(1+s)(1+0.01s)}$$

Step 2 The equivalent sinusoidal transfer function is obtained by replacing s by $j\omega$.

$$G(j\omega) = G(s)\big|_{s=j\omega} = \frac{20}{j\omega(1+j\omega)(1+j0.01\omega)}$$

Step 3 *Identify factors.* The factors of the sinusoidal transfer function in order of increasing frequency are as follows :

- Constant gain $K = 20$
- Pole at the origin: origin $\frac{1}{j\omega}$
- Factor $\frac{1}{1+j\omega}$, pole at $s = -1$
- Factor $\frac{1}{1+j0.01\omega}$, pole at $s = -100$

Step 4 Find the corner frequencies for all factors.

- For the factor 20, the corner frequency is none.
- For the factor $\frac{1}{j\omega}$, the corner frequency is none.
- For the factor $\frac{1}{1+j\omega}$, the corner frequency is 1 rad/s
- For the factor $\frac{1}{1+j0.01\omega}$, the corner frequency is 100 rad/s

Step 5 *Magnitude plot.* For magnitude plot make the Table E8.5(a).

Table E8.5(a)

Factor	*Corner frequency rad/s*	*Asymptotic log-magnitude characteristic*
$K = 20$	None	The magnitude in dB is $20\log K = 20\log 20 = 26$ dB. This is represented by a horizontal line of slope 0 dB/decade, that is parallel to the ω axis and starting from 26 dB
$\frac{1}{j\omega}$	None	Straight line of constant slope −20 dB/decade through $\omega = 1$ rad/s
$\frac{1}{1+j\omega}$	$\omega_1 = 1$	Straight line of constant slope −20 dB/decade and originating from $\omega_1 = 1$ rad/s
$\frac{1}{1+j0.01\omega}$	$\omega_2 = 100$	Straight line of constant slope −20 dB/decade and originating from $\omega_2 = 100$ rad/s

Draw lines of slope −20 dB/decade, −40 dB/decade, and −60 dB/decade at the corner of the semilog paper as shown in Fig. 8.19.

Mark a point A of magnitude 26 dB at ω = 1 and draw a straight line of slope −20 dB/decade passing through 26 dB point. This line will continue till the next factor becomes dominant at corner frequency $\omega_1 = 1$ rad/s. Hence, the resultant slope from $\omega_1 = 1$ rad/s onwards will be $-20 + (-20) = -40$ dB/decade.

Therefore, from first corner frequency $\omega_1 = 1$ to second corner frequency $\omega_2 = 100$, draw a line having a slope of −40 dB/decade.

The resultant slope from second corner frequency onwards will be

$$[-20 + (-20) + (-20)] = -60 \text{ dB/decade.}$$

Therefore, draw a line of slope −60 dB/decade from the second corner frequency of $\omega_2 = 100$.

This will continue upto $\omega \to \infty$ since there is no other factor present in G(s).

Step 6 *Phase Angle Plot*

$$G(j\omega) = \frac{20}{j\omega(1 + j\omega)(1 + j0.01\omega)}$$

The resultant phase angle is given by

$$\phi = \angle G(j\omega)$$

$$= \angle 0° + \angle \frac{1}{j\omega} + \angle \frac{1}{1 + j\omega} + \angle \frac{1}{1 + j0.01\omega}$$

$$\phi = 0° - 90° - \tan^{-1}\omega - \tan^{-1}(0.01)\omega$$

The values of phase angle for different frequencies are given in Table E8.5(b).

Table E8.5(b)

ω	$\angle \frac{1}{j\omega}$	$-\tan^{-1}\omega$	$-\tan^{-1} 0.01\omega$	φ
0.1	−90°	−5.7°	−0.057°	−95.75°
1.0	−90°	−45°	−0.57°	−135.57°
5	−90°	−71.56°	−1.72°	−163.28°
7	−90°	−78.?°	−2.?°	−171.0°
10	−90°	−84.29°	−5.7°	−180°
50	−90°	−88.85°	−26.56°	−205.41°
∞	−90°	−90°	−90°	−270°

Mark the above points on the semilog paper and draw a smooth curve joining all points to get the phase-angle plot.

The complete magnitude and phase-angle Bode plots are shown in Fig. 8.19.

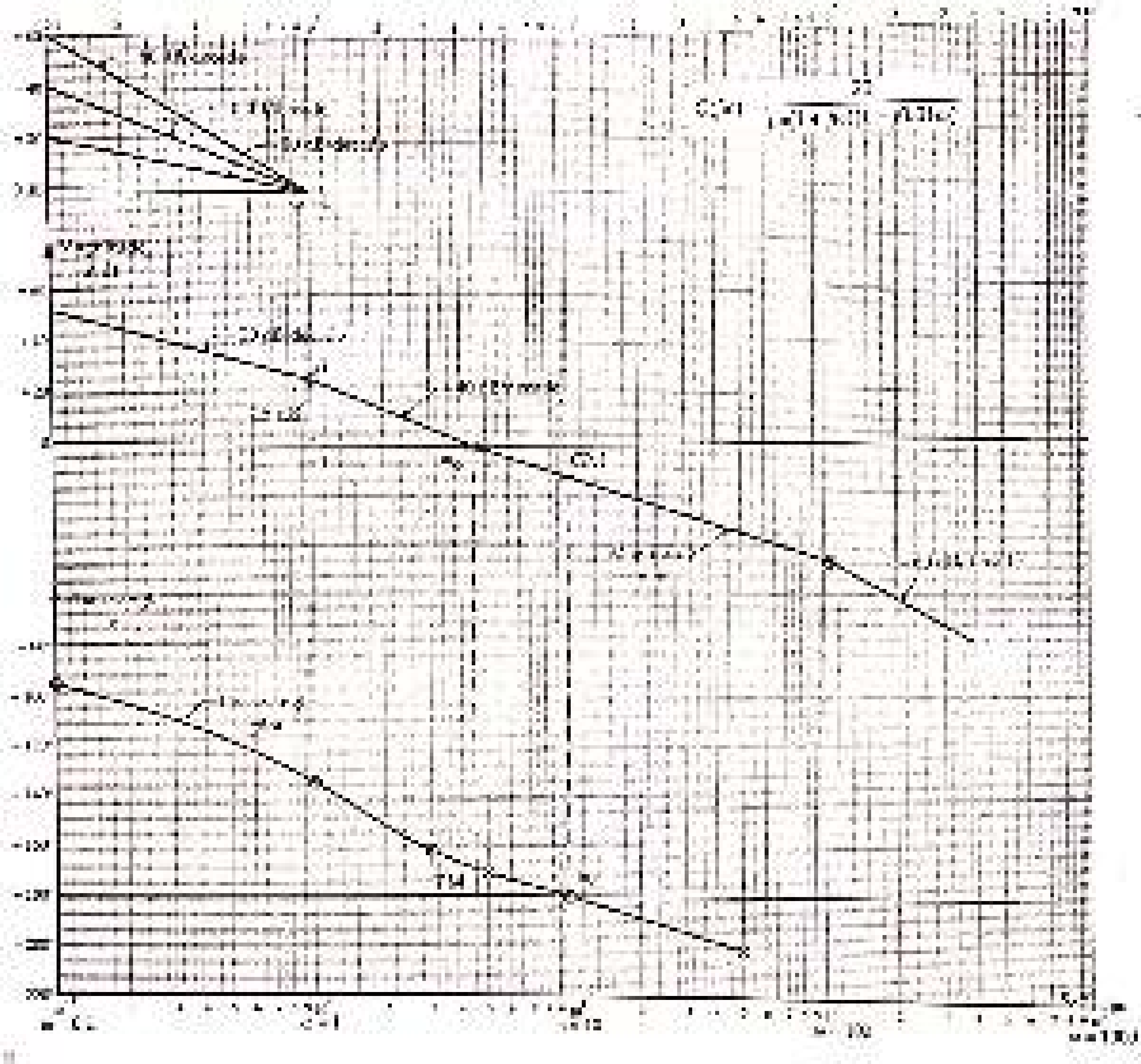


Fig. 8.19

Gain Crossover Frequency ω_{gc} and Phase Margin PM

To determine the gain crossover frequency ω_{gc}, find the point at which the magnitude plot crosses the 0 dB line. From the plot it is found that ω_{gc} = 4.5 rad/s.

From this point draw a vertical line upto the phase angle plot which cuts the phase curve at −172°. Therefore, the phase margin is

$$PM = 180^\circ - 172^\circ = +8^\circ$$

Phase Crossover Frequency ω_{pc} and Gain Margin GM

To determine the phase crossover frequency ω_{pc}, find the point at which the phase angle plot crosses the 180° line. From the plot it is found that:

$$\omega_{pc} = 10 \text{ rad/s}$$

At this frequency draw a vertical line upto the magnitude curve.

This cuts the magnitude curve at magnitude -14 dB. Therefore, the gain margin is 14 dB.

Stability

Since both phase margin and gain margin are positive, the system is absolutely stable.

All pass and minimum phase systems—

(*) A transfer function which has got its poles and zeros only in the left-half of the s-plane is called a "minimum-phase transfer function".

(*) A system whose transfer function is of minimum phase type is called a minimum-phase function.

(*) If all the zeros of a transfer function lie in the right-half of s-plane, all poles lie in the left half of the s-plane, and for every pole in the left half there is a zero in the mirror-image position in the right-half, then the transfer function is called an "all-pass transfer function".

(*) A transfer function which has one or more zeros in the right-half of the s plane is called "non minimum phase transfer function".

(*) A non-minimum-phase transfer function can be treated as a combination of a minimum phase transfer function and an all pass transfer function.

Nyquist Stability Criterion

Relationship between Poles and Zeros of open loop & closed loop transfer function

N = Numerator
D = Denominator

Let the OLTF $= G(s) = \frac{K N(s)}{D(s)}$ and $H(s) = 1$

OLTF = open loop transfer function

Roots of numerator → Zeros of OLTF --- ①
Roots of denominator → Poles of OLTF --- ②

The CLTF of above OLTF G(s)

CLTF = closed loop T/F

$$\frac{G(s)}{1+G(s)H(s)} = \frac{K N(s)}{D(s)+K N(s)}$$

Roots of numerator → Zeros of CLTF --- ③
Roots of denominator → Poles of CLTF --- ④

Characteristic eqn of above OLTF $G(s) = \frac{K N(s)}{D(s)}$ is

$$CE = 1+G(s)H(s) = 1+\frac{K N(s)}{D(s)} = \frac{D(s)+K N(s)}{D(s)}$$

Roots of numerator → Zeros of characteristic equation --- ⑤
Roots of denominator → Poles of characteristic equation --- ⑥

From above six statements we can conclude

(i) poles of characteristic eqn = poles of open loop T/F.

(ii) Zeros of characteristic eqn = poles of closed loop T/F.

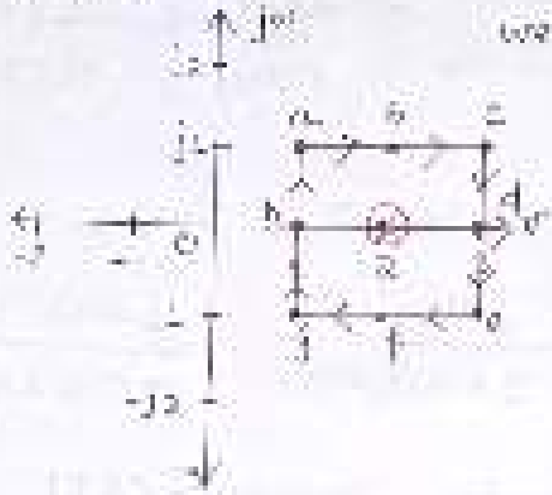
Mapping Theorem or Principle of argument

Concept :—

Let one open loop transfer function be

$$G(s) = \frac{1}{s-2}$$, where "pole" is at 2 on s plane.

if we will represent the pole in s-plane then the diagram will be like —



we will represent pole like ⊗

Closed Contour :— A closed contour in a complex plane is a continuous curve beginning and ending at the same point.

Now we will make one closed contour around the pole, with the assumption that the direction of the continuous curve will be "clockwise".

'a' will be the starting point of the contour and along the clockwise path a→b→c→d→e→f→g→h→a.

As we have got some points along this contour we will have different expressions for G(s) at different points. Let's make a tabulation of it.

Points	$G(s) = \frac{1}{s-2}$	$\lvert G(s)\rvert$	$\angle G(s)$
$s = a = 1+j1$	$1/(-1+j1)$	$1/\sqrt{2}$	-135°
$s = b = 2+j1$	$1/j1$	1	-90°
$s = c = 3+j1$	$1/(1+j1)$	$1/\sqrt{2}$	-45°
$s = d = 3+j0$	1	1	0°
$s = e = 3-j1$	$1/(1-j1)$	$1/\sqrt{2}$	45°
$s = f = 2-j1$	$1/(-j1)$	1	90°
$s = g = 1-j1$	$1/(-1-j1)$	$1/\sqrt{2}$	135°
$s = h = 1+j0$	$1/(-1)$	1	180°

$|G(s)|$ at $s = a$, $G(s) = \dfrac{1}{-1+j1}$

$$|G(s)| = \frac{1}{\sqrt{(-1)^2+(1)^2}} = \frac{1}{\sqrt{2}}$$

$\angle G(s)$ at $s = a$, $G(s) = \dfrac{1}{-1+j1}$

As we can see $-1+j1$ is in second quadrant
the angle must be in second quadrant.

$$\tan^{-1}\frac{1}{-1} = -45^\circ$$

here the angle is coming -45° (4^{th} quadrant)
Hence we have to add 180° to find out the angle
in 2^{nd} quadrant

$$-45^\circ + 180^\circ = 135^\circ$$

Scanned by CamScanner

Now $\angle G(s) = \dfrac{\angle 0^\circ}{\angle 135^\circ} = 0^\circ - 135^\circ = -135^\circ$

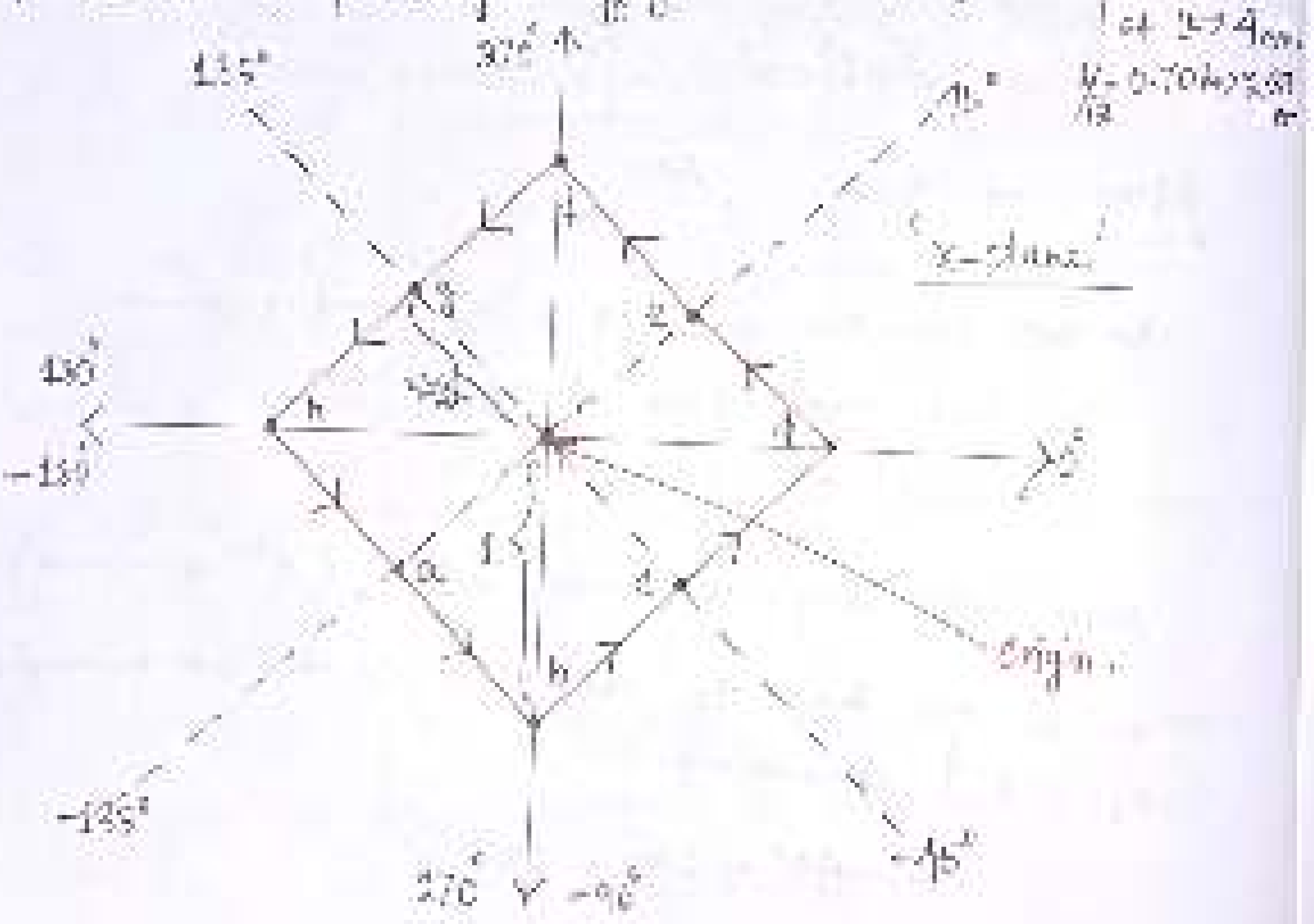
Like this every magnitude and angle can be found.

Mapping Contours in s-plane :-

A contour map is a trajectory in one plane translated in to another plane by a relation F(s).

If we take a complex no. in s-plane and substitute it in to a function F(s) another complex number results. This process is called 'Mapping'.

From the tabulation we can map s-plane in to another plane let 'x' plane by forming [illegible]



Observation —

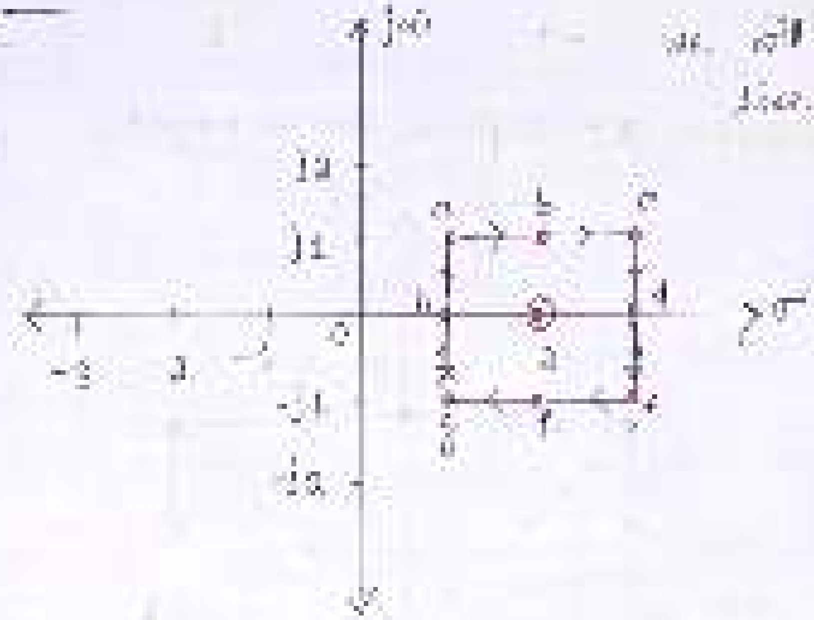
(i) In S-plane the contour encircled the pole with clockwise direction.

(ii) In x-plane the contour encircled the origin with anticlockwise direction.

Next another open loop transfer function

$G(s) = s-2$, where 'zero' is at $+2$ in s-plane.

If we will represent the 'zero' in s-plane then the figure will be like —



we will represent zero like ⊙

Now we will make one closed contour around the zero, with the assumption that the direction of encirclement will be "clockwise".

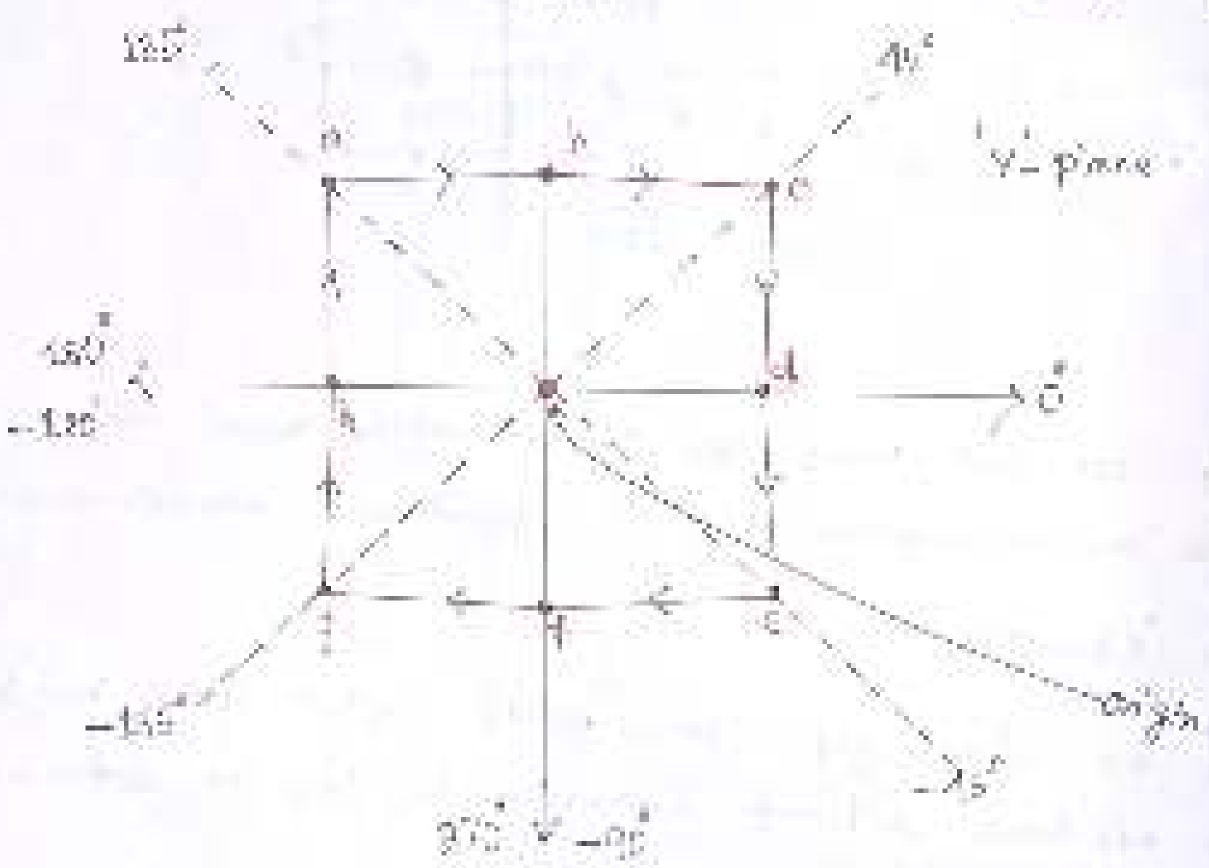
As we have got some points a to h from this contour, we will have different expressions for $G(s)$ at different points. Let's make a tabulation of these values.

Points	$G(s) = s-2$	$\lvert G(s)\rvert$	$\angle G(s)$
$s=a = 1+j1$	$-1+j1$	$\sqrt{2}$	135°
$s=b = 2+j1$	$j1$	1	90°
$s=c = 3+j1$	$1+j1$	$\sqrt{2}$	45°
$s=d = 3+j0$	1	1	0°
$s=e = 3-j1$	$1-j1$	$\sqrt{2}$	-45°
$s=f = 2-j1$	$-j1$	1	-90°
$s=g = 1-j1$	$-1-j1$	$\sqrt{2}$	-135°
$s=h = 1+j0$	-1	1	-180°

From the tabulation the s-plane are is mapped.

$1 \text{ cm} = 1 \Rightarrow 3 \text{ cm}$

$1.414 \times 3 \Rightarrow 4.2 \text{ cm}$



Observation—

* In s-plane the contour encircled the zero with clockwise direction.
* In Y-plane the contour encircled the origin with clockwise direction.

What we noticed from above two mapping is

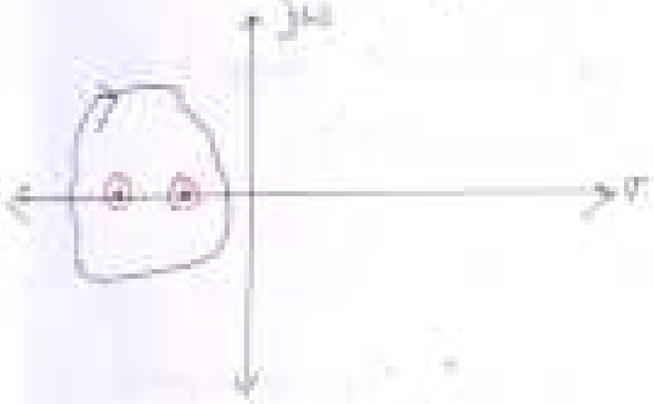
Poles oppose the encirclement

Zeros support the encirclement

Here in both the cases we have taken one pole and one zero. but it is not the case all the time.

for an example, let one open loop transfer function has only a two zeros on the left half of s-plane.

then if we represent those two zeros in s-plane then,



Here we have taken the encirclement about the zeros in clockwise direction and it's not necessary that encirclement is always square in shape. It may be of any shape but it should be a continuous path having starting and ending point same.

Scanned by CamScanner

Now we want to prove that if we do the mapping then how it will look like.

When ever more than one pole and zero will be there then in order to find out the no. of encirclement, we have a formula

$$\boxed{N = P - Z}$$

where P = No. of poles of the CLTF
Z = No. of zeros of the CLTF
N = No. of encirclement about origin.

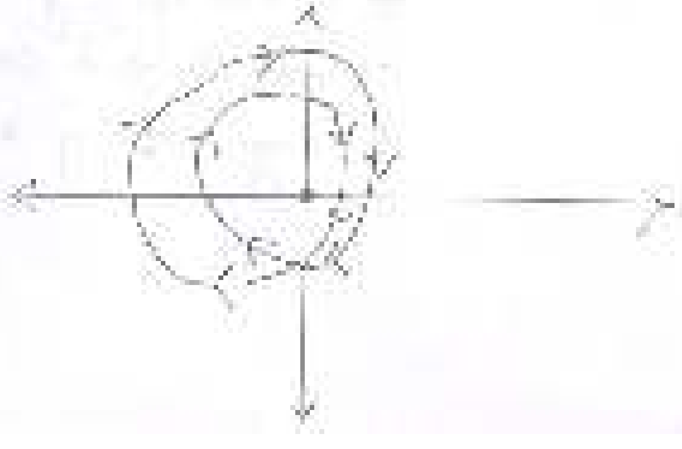
If P = Z then N = 0, hence no encirclement about origin.

The direction of encirclement depends on the greater value of P and Z.

* If P is greater then encirclement oppose the original direction. (N = +ve, anti clockwise)
* If Z is greater then encirclement support the original direction. (N = -ve, clockwise)

In the previous example P = 0, Z = 2.

N = P - Z = 0 - 2 = -2. (It will support the original direction)



From the above content we can conclude that,

Let $F(s)$ be a function that has a finite no. of zeros, poles in the s plane. Suppose that an arbitrary closed path C is chosen in the s-plane so that the path C does not go through any of the poles or zeros of F(s), the corresponding C' locus mapped in the F(s) plane will encircle the origin as many times as the difference between the number of zeros and poles of F(s) that are encircled by the s-plane locus C.

This above theory is called "principle of argument"

Nyquist path or Nyquist Contour

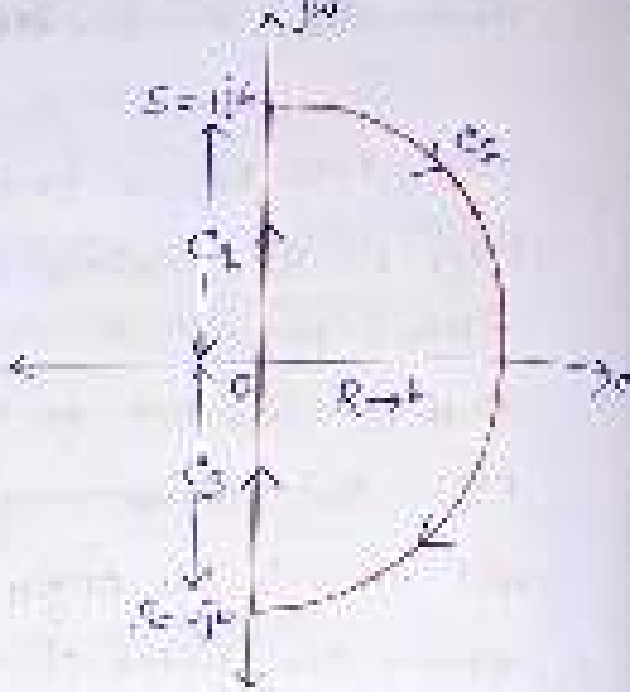
For stability of a closed loop system, the zeros of the characteristic equation

$$F(s) = 1 + G(s)H(s) = 0$$

must lie in the left-hand side of the imaginary axis in the s plane (as we have already proved that zeros of characteristic eq^n = poles of closed loop transfer function). If any one zero of $1+G(s)H(s)$ lie in the right half of the s-plane then the system is unstable.

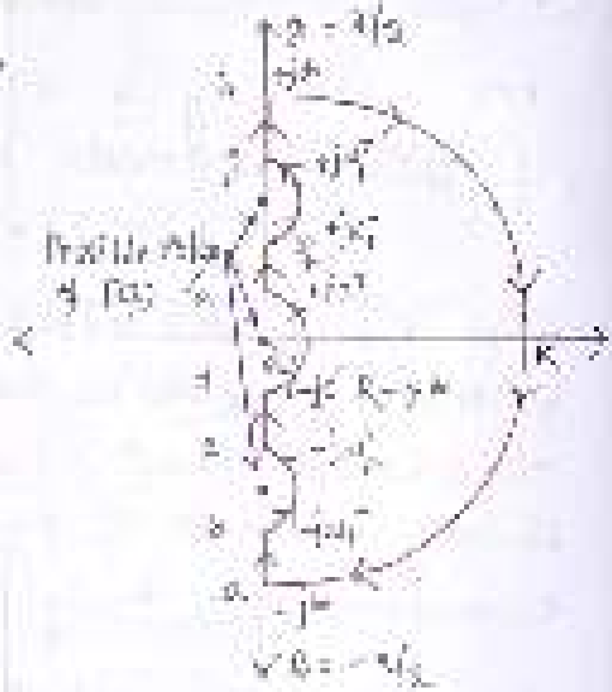
Nyquist had suggested that rather than analyzing whether all the zeros are located in the left half of the s-plane, it is better to examine the presence of any one zero of $1+G(s)H(s)$ in the right of the s-plane making the system unstable.

The Nyquist path or Nyquist contour is so chosen that it encloses the entire right half of the s-plane. It encloses all the right half s-plane poles and zeros of $G(s)H(s)$. This Nyquist path is composed of three segments C_1, C_2 and C_3. The first segment is the half imaginary axis from 0 to $+j\infty$. The segment C_2 is a semicircle of radius infinity that encloses the entire right half of the s-plane. The segment C_3 the negative half imaginary axis from $-j\infty$ to 0.



According to principle of argument, the Nyquist path should not pass through the singular points of $F(s)$ i.e. the poles at the origin or poles on the $j\omega$ axis of the open loop transfer function.

If the $G(s)H(s)$ has poles at the origin or poles on the $j\omega$ axis at points other than the origin, the contour in the s-plane must be modified so as to bypass any $j\omega$ axis pole. This can be done by selecting a small semicircle of radius $\varepsilon \to 0$.



When drawing the nyquist contour, we will divide the whole contour into small parts i.e.,

ab : from $\omega = -\infty$ to $\omega = -\omega_1^-$, θ varies from $-\pi/2$ to $+\pi/2$.

- bc : from $\omega = -\omega_1^-$ to $\omega = -\omega_1^+$, θ varies from $-\pi/2$ to $\pi/2$.

cd : from $\omega = -\omega_1^+$ to $\omega = 0^-$, θ varies from $-\pi/2$ to $\pi/2$

- de : from $\omega = 0^-$ to $\omega = 0^+$, θ varies from $-\pi/2$ to $\pi/2$

ef : from $\omega = 0^+$ to $\omega = \omega_1^-$, θ varies from $-\pi/2$ to $\pi/2$.

- fg : from $\omega = \omega_1^-$ to $\omega = \omega_1^+$, θ varies from $-\pi/2$ to $\pi/2$.

gh : from $\omega = \omega_1^+$ to $\omega = \infty$, θ varies from $-\pi/2$ to $\pi/2$

Here the semicircle bc, de, fg are

$\lim\limits_{r \to 0} re^{j\theta}$, where θ varies from $-\pi/2$ to $\pi/2$.

$re^{j\theta}$ - equation of a circle whose radius is r.

- hka : from $\omega = +\infty$ to $\omega = -\infty$, θ varies from $\pi/2$ to $-\pi/2$.

$\lim\limits_{R \to \infty} Re^{j\theta}$, where θ varies from $\pi/2$ to $-\pi/2$.

Nyquist stability Criterion:

$$\boxed{N = P - Z}$$

where P = No. of poles of $G(s)H(s)$ (or) $1+G(s)H(s)$ in the right-half of the s-plane.

N = No. of anticlockwise encirclement of origin of $[1+G(s)H(s)]$ plane by the nyquist path.

(or)
No. of anticlockwise encirclement of $(-1, j0)$ point of $G(s)H(s)$ plane by the nyquist path.

Z = No. of zeros of $1+G(s)H(s)$ in the right half of the s-plane.

(or) No. of poles of closed loop transfer function in the right half of the s-plane.

Note:- (i) In some place $\boxed{N = Z - P}$ is also given.
In that case we will go for clockwise encirclement.

(ii) $(-1, j0)$ point is also called critical point.

Hence Nyquist Stability Criterion can be stated as:

If the contour C_{GH} of the open loop transfer function $G(s)H(s)$ in the $G(s)H(s)$ plane corresponding to a Nyquist contour in the s plane encircles the point $(-1, j0)$ in the anticlockwise direction as many times as the right half s-plane poles of $G(s)H(s)$, then the closed loop system is stable.

(or) We can say if $Z = 0$, then CLTF is stable.
If $P = 0$, then OLTF is stable.

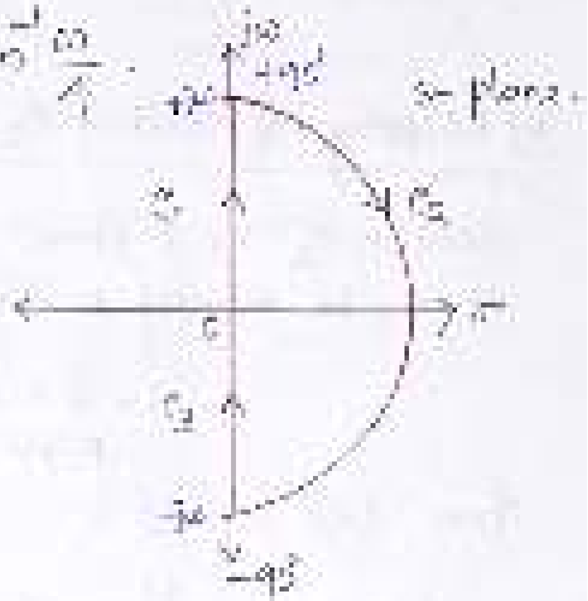
Ques:— Sketch the Nyquist plot for the open-loop transfer function and determine the stability of the closed-loop system by Nyquist stability criterion. $G(s)H(s) = \frac{15}{(s+2)(s+4)}$

Sol:— (a) $G(s)H(s)\Big|_{s=j\omega} = \frac{15}{(j\omega+2)(j\omega+4)}$

$$= \frac{15\angle 0^\circ}{\left(\sqrt{4+\omega^2}\,\angle\tan^{-1}\frac{\omega}{2}\right)\left(\sqrt{16+\omega^2}\,\angle\tan^{-1}\frac{\omega}{4}\right)}$$

hence $M = \frac{15}{\sqrt{4+\omega^2}\sqrt{16+\omega^2}}$

and $\phi = 0^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$.



Step-1

The Nyquist path is shown.

The Nyquist path is divided into three sections C_1, C_2 and C_3. We will map each section from s-plane to GH plane.

Step-2

(i) Mapping of section C_1 — here ω varies from 0^+ to $+\infty$.

At $\omega = 0^+$; $M = \lim_{\omega\to 0}|G(s)H(s)| = \frac{15}{2\times 4} = 1.875$

$$\phi = \lim_{\omega\to 0}\angle\left(-\tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}\right)$$

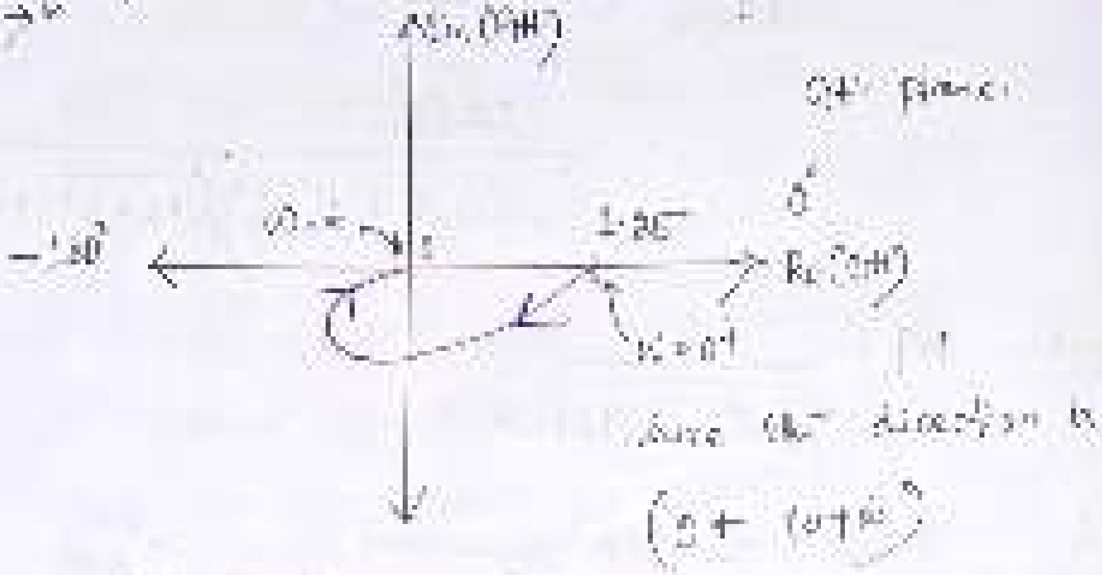
$$= 0^\circ - 0^\circ = 0^\circ$$

The result of this mapping is the polar plot of the $G(s)H(s)$.

At $\omega = +\infty$

$$M_1 = \lim_{\omega\to\infty} \left| G(j\omega)H(j\omega) \right| = \frac{10}{\infty} = 0$$

$$\phi_1 = \lim_{\omega\to\infty} \left(-\tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4} \right) = -90^\circ - 90^\circ = -180^\circ$$



b) Mapping of section C_2 :-

Section C_2 in s plane is a semicircle of infinite radius. It can be mapped in G(s)H(s) plane by substituting

$s = \lim_{R\to\infty} Re^{j\theta}$ in G(s)H(s) and θ will vary from $+90^\circ$ to -90°.

$$G(s)H(s) = \frac{10}{(Re^{j\theta}+2)(Re^{j\theta}+4)}$$

As $R\to\infty$, $Re^{j\theta} \gg 2$ and $Re^{j\theta} \gg 4$. Hence they can be treated as $Re^{j\theta}$ and $Re^{j\theta}$.

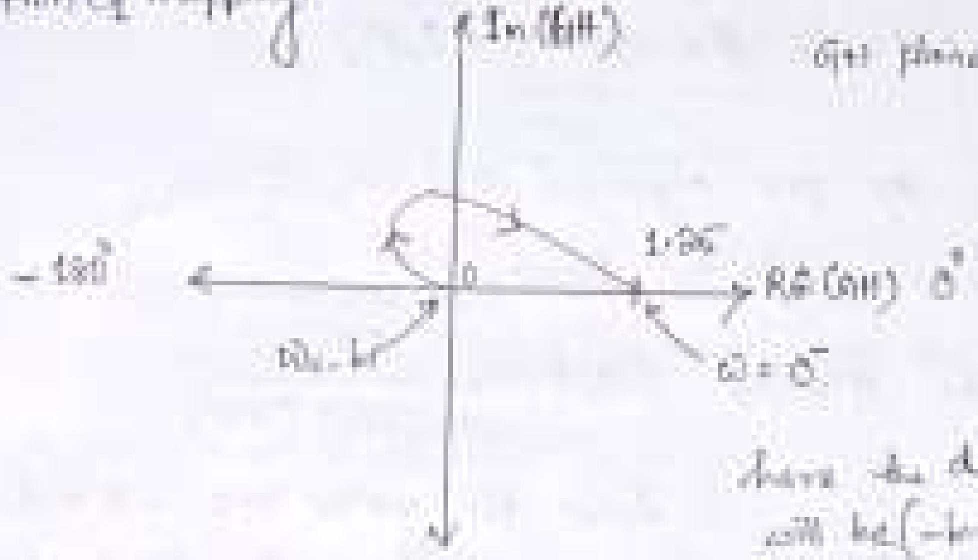
$$\lim_{R\to\infty} G(s)H(s) = \lim_{R\to\infty} \frac{10}{R^2 e^{j2\theta}} = 0\, e^{-j2\theta}$$

Thus the entire semicircle C_2 in s-plane is mapped into a circular arc of zero radius around the origin in the GH plane.

Scanned by CamScanner

c) Mapping of section C_3—

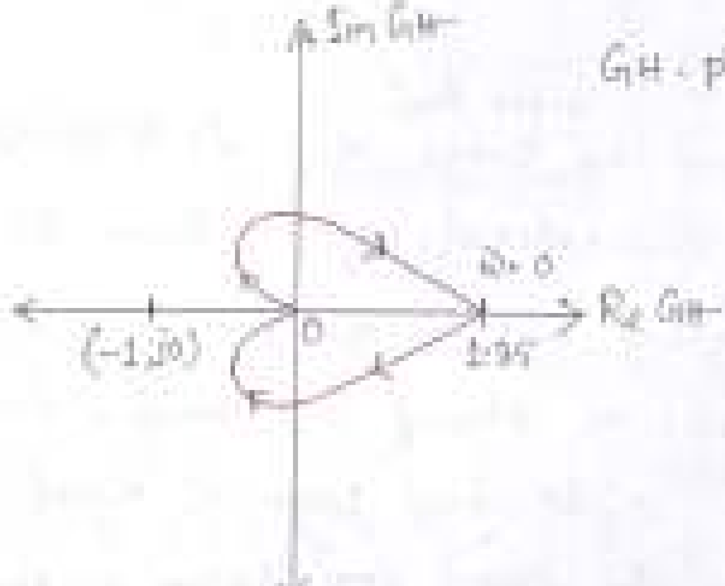
In this section ω varies from $-\infty$ to 0^-. The mapping will be mirror image about the real axis in GH plane corresponding to section C_1. We can say this will be the inverse polar plot of section C_1 mapping.



here the direction will be $(-\infty$ to $0^-)$

Step-3

By combining all mapping result in GH plane we can get the complete Nyquist plot.



Note:— The mapping of section C_1 in GH-plane is nothing but the polar plot and the mapping of C_3 is Inverse polar plot.

Scanned by CamScanner

(ii) Determination of stability –

To determine stability, we need to know whether the Nyquist plot encircles the critical point (-1, j0) or not.

From the complete Nyquist plot we can see Nyquist plot doesn't encircle the point (-1, j0)

Therefore as per Nyquist stability criteria –

$N = 0$ here.

From the equation $G(s)H(s) = \frac{10}{(s+2)(s+4)}$

here the poles are -2 and -4

both these poles are on left half of s-plane.

Since there are no poles of G(s)H(s) in the right half of s-plane

$P = 0$ here.

as per encirclement criterion $N = Z - P$

$\Rightarrow 0 = Z - 0$

$\Rightarrow Z = 0$.

Thus there are no right half zeros of 1+G(s)H(s) which is nothing but characteristic eqn. From the beginning we have proved that zeros of characteristic eqn is same as poles of closed loop transfer function. Hence there is no right half poles of closed loop transfer function. Hence the closed loop system is stable.

Q:- The open-loop transfer function of a unity feedback system is given by $G(s)H(s) = 50/s(s+5)$. Draw the Nyquist plot and comment on the stability of the closed loop system.

Solⁿ:— Here the open-loop system given is

$$G(s)H(s) = \frac{50}{s(s+5)}$$

In this OLTF one pole is at origin. According to principle of argument, we have to bypass the origin while drawing Nyquist path.

The given OLTF in sinusoidal form is

$$G(j\omega)H(j\omega) = \frac{50}{j\omega(j\omega+5)}$$

$$|G(j\omega)H(j\omega)| = \frac{50}{\omega\sqrt{25+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle 0^\circ}{\angle 90^\circ + \angle \tan^{-1}\frac{\omega}{5}} = -90^\circ - \tan^{-1}\frac{\omega}{5}$$

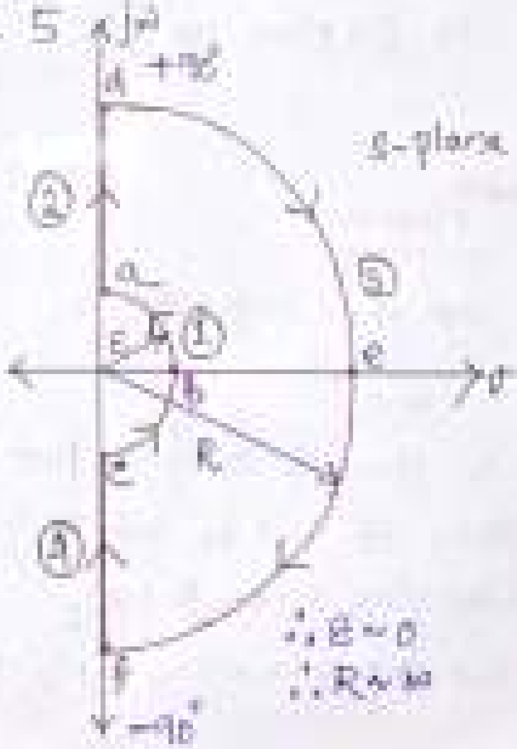
The nyquist path is drawn and the nyquist path is divided in to four sections.

i) semicircle abc

ii) segment cd on Imaginary axis

iii) semicircle def.

iv) segment fa on imaginary axis.



iii) Mapping of semicircle abc

This semicircle in s-plane can be mapped in G(s)H(s) plane by substituting $s = \lim_{\epsilon \to 0} \epsilon e^{j\theta}$, θ varying from $-\pi/2$ to $\pi/2$.

$$G(s)H(s)\Big|_{s=\epsilon e^{j\theta}} = \frac{50}{\epsilon e^{j\theta}(\epsilon e^{j\theta}+5)}$$

$$\lim_{\epsilon \to 0} G(\epsilon e^{j\theta})H(\epsilon e^{j\theta}) = \lim_{\epsilon \to 0} \frac{50}{\epsilon e^{j\theta}(\epsilon e^{j\theta}+5)}$$

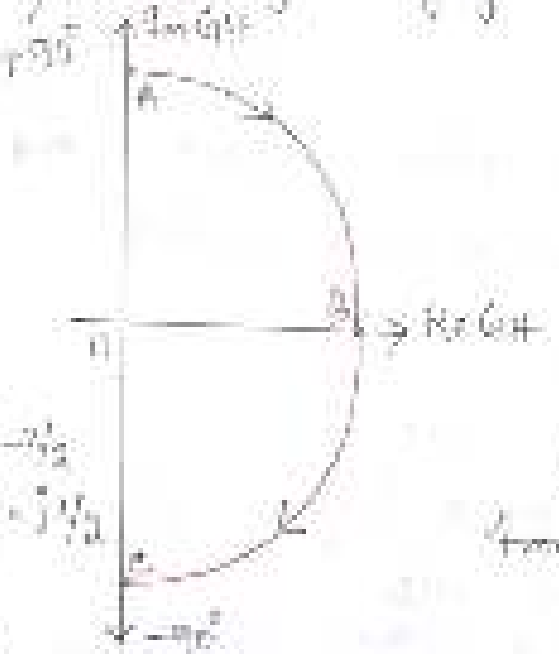
$$= \lim_{\epsilon \to 0} \frac{50}{\epsilon^2 e^{j2\theta} + 5\epsilon e^{j\theta}}$$

$$= \lim_{\epsilon \to 0} \frac{50}{5\epsilon e^{j\theta}}$$

$$= \lim_{\epsilon \to 0} \frac{10 e^{-j\theta}}{\epsilon}$$

$$= \infty e^{-j\theta}$$

Hence abc semicircle in s plane is mapped in to a semicircle of ∞ radius with angle varying from $\pi/2$ to $-\pi/2$ in GH plane as ABC.



Note--

as the expression is $\infty e^{-j\theta}$

In place of θ if we put $-\pi/2$ then it will be $-(-j\pi/2) = j\pi/2$

Likewise when $\theta = \pi/2$ the expression will be $-(j\pi/2)$

direction of θ from $+\pi/2$ to $-\pi/2$

ii) Mapping of segment cd

Here ω varies from $\omega = 0^+$ to $\omega = +\infty$.

At $\omega = 0^+$
$$G(j\omega)H(j\omega) = \frac{50}{j0(j0+5)}$$

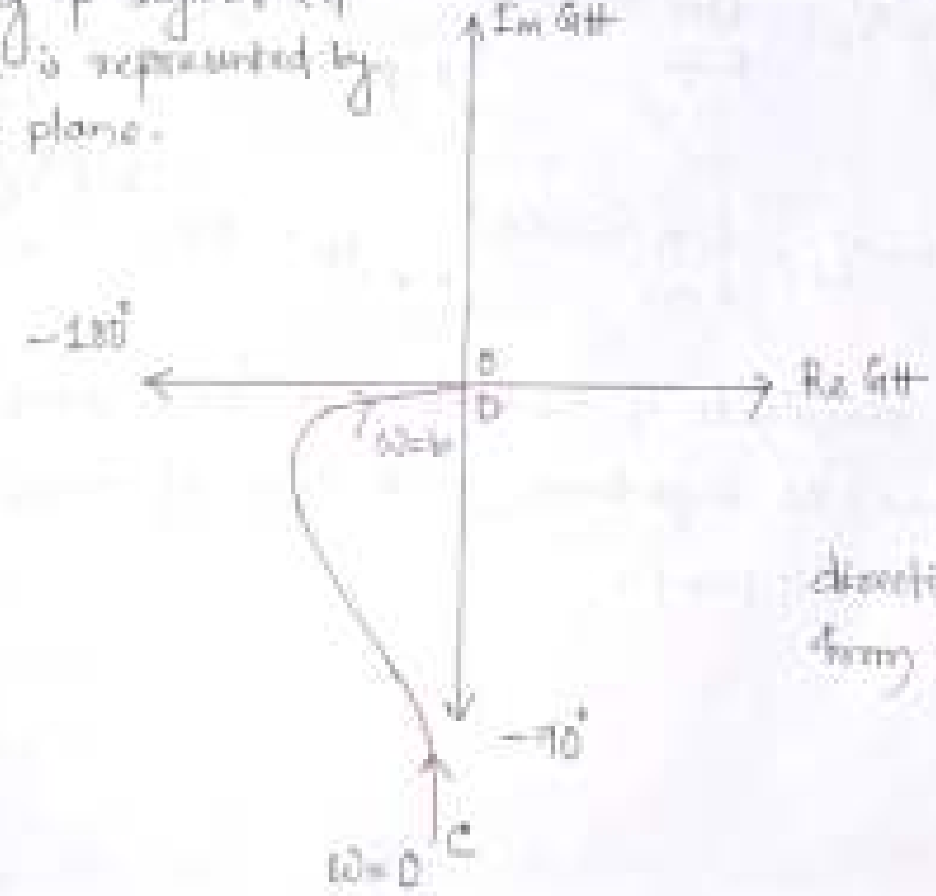
$$|G(j\omega)H(j\omega)| = \frac{\sqrt{50^2}}{\sqrt{0^2} \times \sqrt{5^2}} = \infty .$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle 0^\circ}{90^\circ + \tan^{-1}\cancel{\frac{0}{5}}\,0^\circ} = -90^\circ .$$

At $\omega = \infty$
$$|G(j\omega)H(j\omega)| = \frac{\sqrt{50^2}}{\sqrt{\infty^2} \times \sqrt{\infty^2+5^2}} = 0 .$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle 0^\circ}{90^\circ + \tan^{-1}\frac{\infty}{5}} = -180^\circ .$$

The mapping of segment cd in s-plane is represented by CD in GH plane.



direction will be from $\omega = 0$ to $\omega = \infty$

Scanned by CamScanner

iii) Mapping of semicircle def

The semicircle def is represented by $s = Re^{j\theta}$ with $R \to \infty$ and θ varying from $+90^\circ$ through 0° to -90°.

$$G(s)H(s) = \frac{50}{Re^{j\theta}(Re^{j\theta}+5)}$$

As $R \to \infty$, $Re^{j\theta} \gg 5$, hence $Re^{j\theta}+5$ is approximated to $Re^{j\theta}$

$$\lim_{R\to\infty} \frac{50}{Re^{j\theta} \cdot Re^{j\theta}}$$

$$= \lim_{R\to\infty} \frac{50}{R^2 e^{j2\theta}}$$

$$= \lim_{R\to\infty} \frac{50\, e^{-j2\theta}}{R^2}$$

$$= 0\, e^{-j2\theta}$$

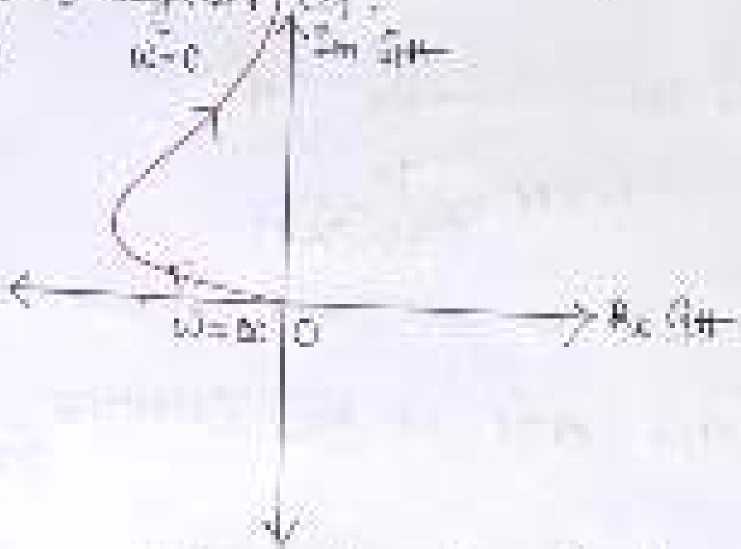
When $\theta = \pi/2$, $\lim_{R\to\infty} G(s)H(s)\Big|_{s=Re^{j\theta}} = 0e^{-j2\times\pi/2} = 0e^{-j\pi}$

When $\theta = -\pi/2$, $\lim_{R\to\infty} G(s)H(s)\Big|_{s=Re^{j\theta}} = 0e^{-j2\times(-\pi/2)} = 0e^{j\pi}$

As the magnitude of the vector is zero here, so the angle has no importance. A circle of zero radius is a single 'def' point.

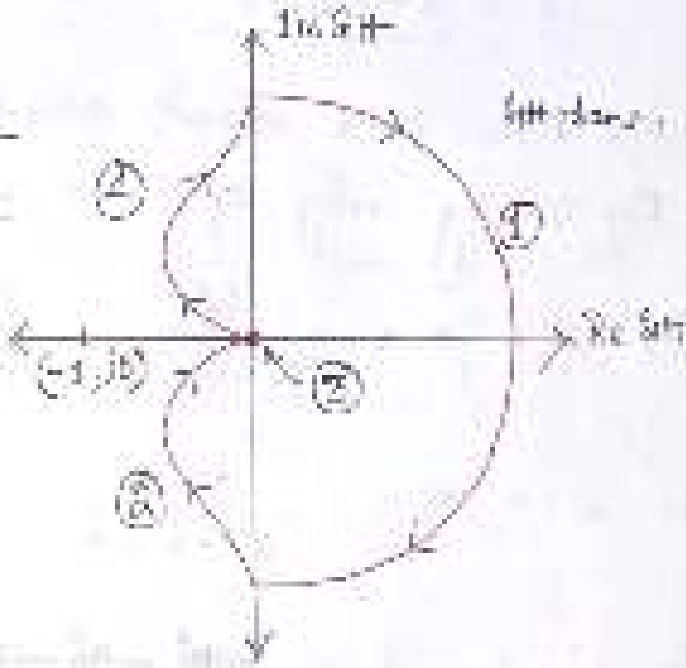
iv) Mapping of segment fa

In this section ω varies from -∞ to 0⁻. The mapping will be mirror image about the real axis in GH plane corresponding to segment cd.



In order to get the complete Nyquist plot in the G(s)H(s) plane we add all the mappings in a single G(s)H(s) plane.

It is seen that the number of encirclement N of the point (-1, j0) by GH contour is zero i.e. N = 0.



From this OLTF it is clear that there is no right hand side poles.
i.e. P = 0.

Here the direction is clockwise.

Hence $N = Z - P$

$Z = N + P$

$Z = 0 + 0 = 0$

That means no pole of the CLTF lies on right side hence stable

Effect of addition of poles on shape of Nyquist plot

To investigate this we will take a previously solved example. In that we will add pole and we will see the effect.

Let us take the example of

$$G(s)H(s) = \frac{10}{(s+2)(s+4)}$$

In this OLTF we will add a pole $s = -6$.

So now the OLTF will be $G(s)H(s) = \frac{10}{(s+2)(s+4)(s+6)}$

$$M = \left|G(j\omega)H(j\omega)\right| = \frac{10}{\sqrt{4+\omega^2}\sqrt{16+\omega^2}\sqrt{36+\omega^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4} - \tan^{-1}\frac{\omega}{6}$$

As we have already solved this un-modified question. By solving only ϕ section we can predict the nyquist plot

at $\omega = 0$, $M = \frac{10}{2\times4\times6} = \frac{5}{24} = 0.208$.

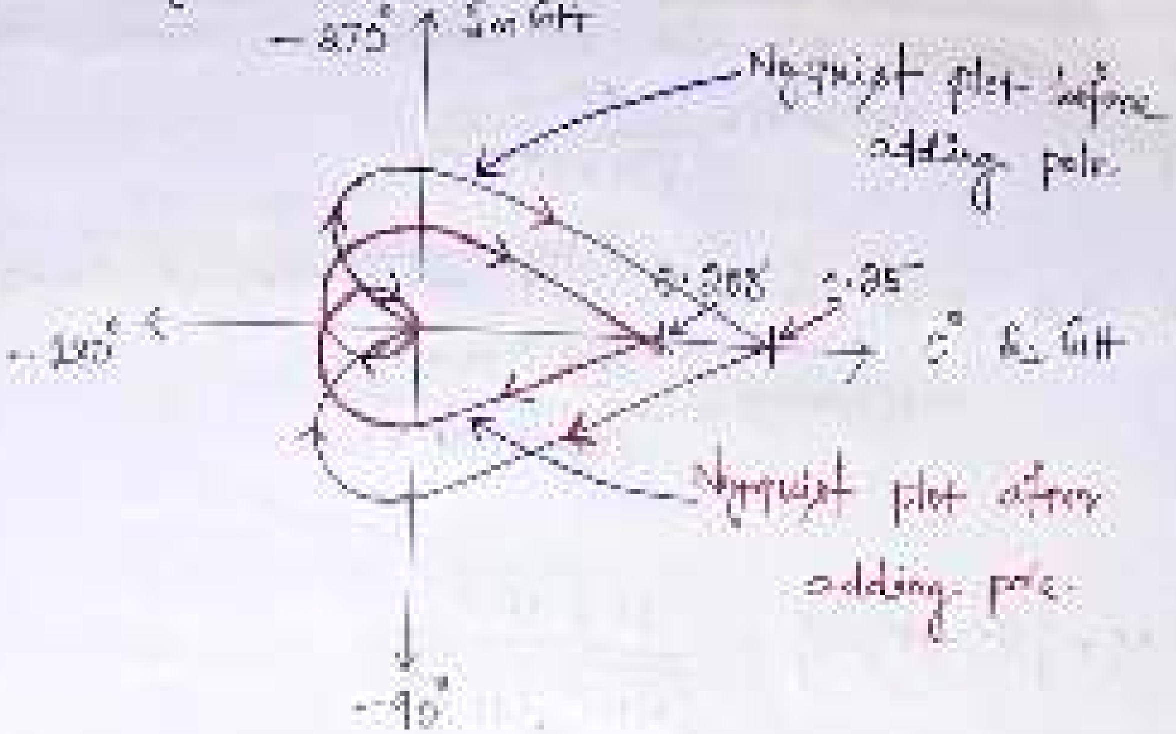
at $\omega = \infty$, $M = \frac{10}{\infty\times\infty\times\infty} = 0$.

at $\omega = 0$, $\phi =$

at $\omega = \infty$, $\phi = -90° - 90° - 90° = -270°$

Scanned by CamScanner

* The complete nyquist plot will be—



OLTF	Magnitude (M)	Angle (θ)
$\frac{10}{(s+2)(s+4)}$	1.25	$-180°$
$\frac{10}{(s+2)(s+4)(s+6)}$	0.208	$-270°$

If we will add one more pole then again we will see reduction in magnitude and shifting of angle $-90°$

Scanned by CamScanner

Effect of addition of zeros on shape of Nyquist plot

To investigate this we will take the same example as that before.

$$G(s)H(s) = \frac{10}{(s+2)(s+4)}$$

In this OLTF we will add one zero at $s=-3$. Now the eqn

$$G(s)H(s) = \frac{10(s+3)}{(s+2)(s+4)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{10\sqrt{9+\omega^2}}{\sqrt{4+\omega^2}\sqrt{16+\omega^2}}$$

$$\phi = \tan^{-1}\frac{\omega}{3} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$$

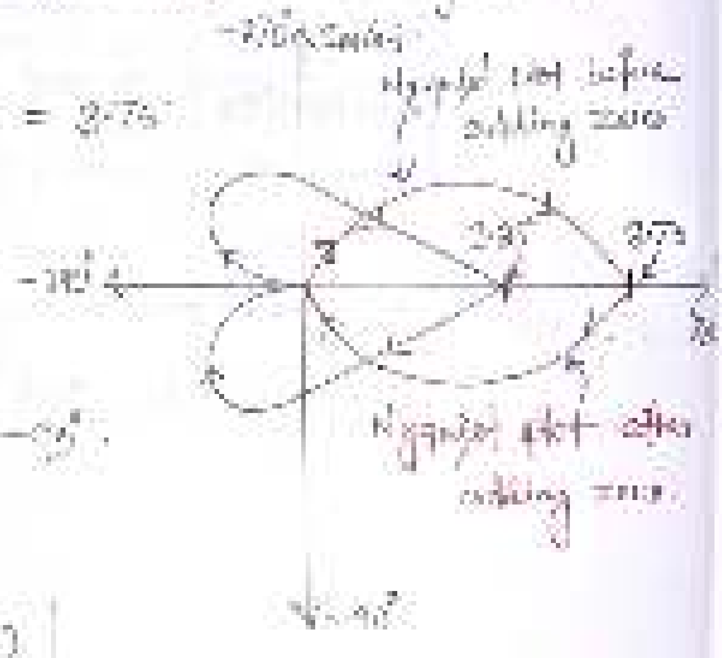
By solving only 1st section we can predict the nyquist plot.

at $\omega=0$, $M = \frac{10\times 3}{2\times 4} = \frac{30}{8} = 3.75$

$\phi = 0^\circ$

at $\omega=\infty$, $M = \frac{10\times\omega}{\omega\times\omega} = 0$

$\phi = 90^\circ - 90^\circ - 90^\circ = -90^\circ$

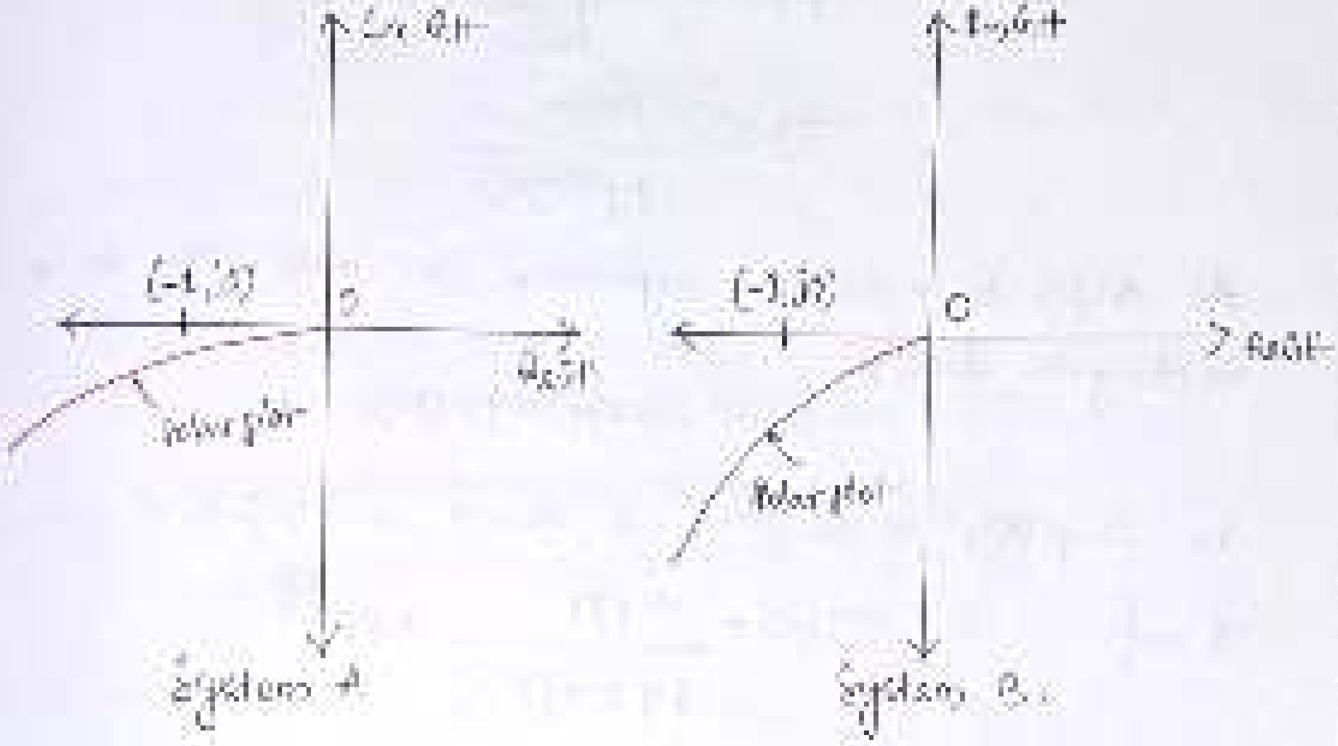


OLTF	Magnitude (M)	Angle (ϕ)
$\frac{10}{(s+2)(s+4)}$	1.25	-180°
$\frac{10(s+3)}{(s+2)(s+4)}$	3.75	-90°

If we will add one more zero again we will see increase in magnitude and shifting of angle $+90^\circ$.

Assessment of relative stability using Nyquist criterion

A measure of relative stability of LTI system is the proximity of $G(j\omega)H(j\omega)$ locus to $(-1, j0)$ point. As the polar plot gets closer to the $(-1, j0)$ point, the relative stability reduces and the system moves towards instability.



Here the polar plot of system A is more closer to (-1, j0) point than system B's polar plot. Hence system B is more stable than system A.

As the polar plot moves closer to the (-1, j0) point, the system closed loop poles moves closer to $j\omega$ axis and hence the system becomes relatively less stable.

Constant M circles (Constant magnitude loci)

Consider a unity feedback system with closed loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)}$$

For sinusoidal transfer function, replace s by $j\omega$.

$$T(j\omega) = T(s)\Big|_{s=j\omega}$$

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)}$$

As $T(j\omega)$ is a complex expression, we can write it in rectangular form:-

$$G(j\omega) = X(j\omega) + jY(j\omega)$$

For simplicity let us take $X(j\omega) = X$ and $Y(j\omega) = Y$.

Therefore, $$T(j\omega) = \frac{X+jY}{1+X+jY} = Me^{j\phi}$$

Where M is the magnitude of $T(j\omega)$

i.e. $M = |T(j\omega)|$

$$M = \frac{|X+jY|}{|1+X+jY|}$$

$$= \frac{\sqrt{X^2+Y^2}}{\sqrt{(1+X)^2+Y^2}}$$

Squaring both sides we get

$$M^2 = \frac{X^2+Y^2}{(1+X)^2+Y^2}$$

$$M^2\left[(1+X)^2+Y^2\right] = X^2+Y^2$$

$$M^2(1+2x+x^2+y^2) - x^2 - y^2 = 0$$

$$x^2(M^2-1) + 2xM^2 + y^2(M^2-1) + M^2 = 0 \quad \text{— (9.2 k)}$$

Dividing by (M^2-1) we get —

$$x^2 + \frac{2xM^2}{M^2-1} + y^2 + \frac{M^2}{M^2-1} = 0$$

$$x^2 + \frac{2xM^2}{-(1-M^2)} + y^2 + \frac{M^2}{-(1-M^2)} = 0$$

$$x^2 - \frac{2xM^2}{(1-M^2)} + y^2 = \frac{M^2}{(1-M^2)}$$

Adding $\left[\frac{M^2}{1-M^2}\right]^2$ on both sides to complete the square on

left hand side we get

$$\underbrace{x^2 - \frac{2xM^2}{(1-M^2)} + \left[\frac{M^2}{(1-M^2)}\right]^2}_{\left[x - \frac{M^2}{1-M^2}\right]^2} + y^2 = \frac{M^2}{1-M^2} + \left[\frac{M^2}{1-M^2}\right]^2$$

$$\left[x - \frac{M^2}{1-M^2}\right]^2 + y^2 = \frac{M^2}{1-M^2}\left[1 + \frac{M^2}{1-M^2}\right]$$

$$\left[x - \frac{M^2}{1-M^2}\right]^2 + y^2 = \frac{M^2}{1-M^2}\left[\frac{1-\cancel{M^2}+\cancel{M^2}}{1-M^2}\right]$$

$$\boxed{\left[x - \frac{M^2}{1-M^2}\right]^2 + y^2 = \left[\frac{M}{1-M^2}\right]^2}$$

$\therefore (x-a)^2 + (y-b)^2 = r^2$

equation of a circle

centre at (a,b)

radius $= r$

This is the equation of a circle on the x-y plane with centre located at $X_0 = \frac{M^2}{1-M^2}$

$$Y_0 = 0$$

The radius of the circle is $r_0 = \frac{M}{1-M^2}$

For different values of M we get the family of circles of different radii and centre in the [G(jω) = X + jY] plane.

For a particular circle, the value of M, i.e. the magnitude of the closed loop transfer function is constant. Hence these circles are called constant magnitude loci or constant M circles.

(a) For M = 1, eq^n gives $2X + 1 = 0$ or $X = -\frac{1}{2}$.
This is an equation of a straight line parallel to y axis and passing through the [illegible]

M	Centre (X_0, Y_0)	Radius (r_0)
0.2	0.04, 0	0.20
0.3	[illegible], 0	0.33
0.8	1.78, 0	2.25
0.9	4.26, 0	4.73
1	NA	NA
1.1	−5.76, 0	|−5.23|
1.2	−3.27, 0	|−2.73|
1.6	−1.64, 0	|−1.02|
2.1	−1.3, 0	|−0.6|

For M > 1 the centres are to the left of $(-\frac{1}{2}, 0)$
For M < 1, the centres are to the right of origin.
At M = 1, the circle degenerates in to straight line.

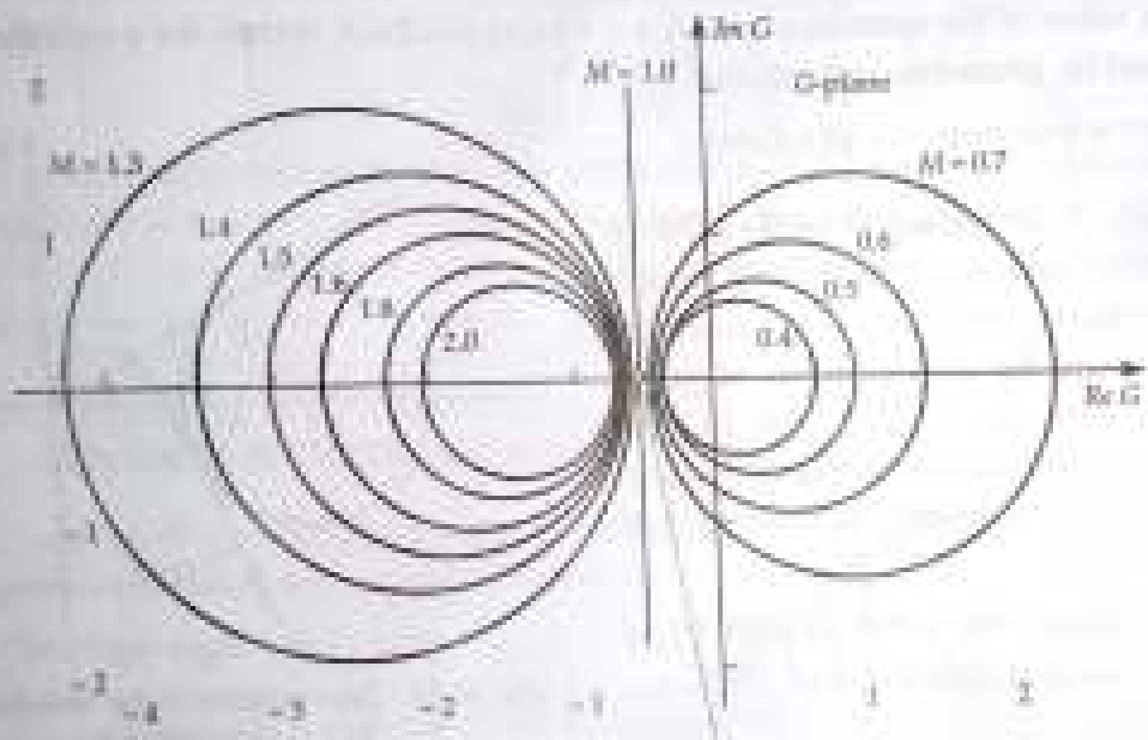


Fig. 11.18 A family of constant-M circles

$(-1/2, 0)$

Constant-N Circles (Constant-Phase Loci)

for the same closed loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)}$$

the sinusoidal t/f will be $T(j\omega) = \dfrac{G(j\omega)}{1+G(j\omega)}$

and according to the complex form

$$T(j\omega) = \frac{X+jY}{1+X+jY}$$

The phase angle ϕ of $T(j\omega)$ is given by

$$\angle T(j\omega) = \phi = \angle\left[\frac{X+jY}{1+X+jY}\right]$$

$$\phi = \tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y}{1+X}$$

Let $\tan\phi = N$

then $N = \tan\left[\tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y}{1+X}\right]$

Since $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \cdot \tan B}$

we get $N = \dfrac{\tan\left[\tan^{-1}\frac{Y}{X}\right] - \tan\left[\tan^{-1}\frac{Y}{1+X}\right]}{1+\tan\left[\tan^{-1}\frac{Y}{X}\right]\cdot\tan\left[\tan^{-1}\frac{Y}{1+X}\right]}$

$$N = \frac{\frac{Y}{X} - \frac{Y}{1+X}}{1+\left(\frac{Y}{X}\right)\left(\frac{Y}{1+X}\right)}$$

By solving this equation we can get

$$N = \frac{Y}{X(1+X)+Y^2}$$

$$X(1+X)+Y^2 = \frac{Y}{N}$$

$$X^2+X+Y^2-\frac{Y}{N}=0$$

Adding $\frac{1}{4}+\frac{1}{(2N)^2}$ to both sides to complete the square on left hand side, we get

$$X^2+X+\frac{1}{4}+Y^2-\frac{Y}{N}+\left(\frac{1}{2N}\right)^2 = \frac{1}{4}+\frac{1}{(2N)^2}$$

$$\boxed{\left(X+\frac{1}{2}\right)^2+\left(Y-\frac{1}{2N}\right)^2 = \frac{1}{4}+\frac{1}{(2N)^2}} \quad \text{— (eqn - a)}$$

This is an equation of a circle with N as parameter. It has centre at $\left(-\frac{1}{2}, \frac{1}{2N}\right)$ and has radius of

$$r = \sqrt{\frac{1}{4}+\frac{1}{(2N)^2}}$$

For different values of N we get a family of N circles. On a particular circle, the value of N that is, phase angle ϕ remaining constant, hence these circles are called constant phase loci or constant-N circles.

(a) Eqn-a is satisfied by (X=0, Y=0) and (X=-1, Y=0) regardless of the value of N. This implies that all the circles pass through two points, that is the origin (0,0) and the point (-1, j0)

(b) The centre of all N-circles lie on $X=-\frac{1}{2}$ line.

(c) For +ve N, the centre lies on the line $X=-\frac{1}{2}$ and above the horizontal axis. For -ve N, the centre lies on the line $X=-\frac{1}{2}$ and below the horizontal axis.

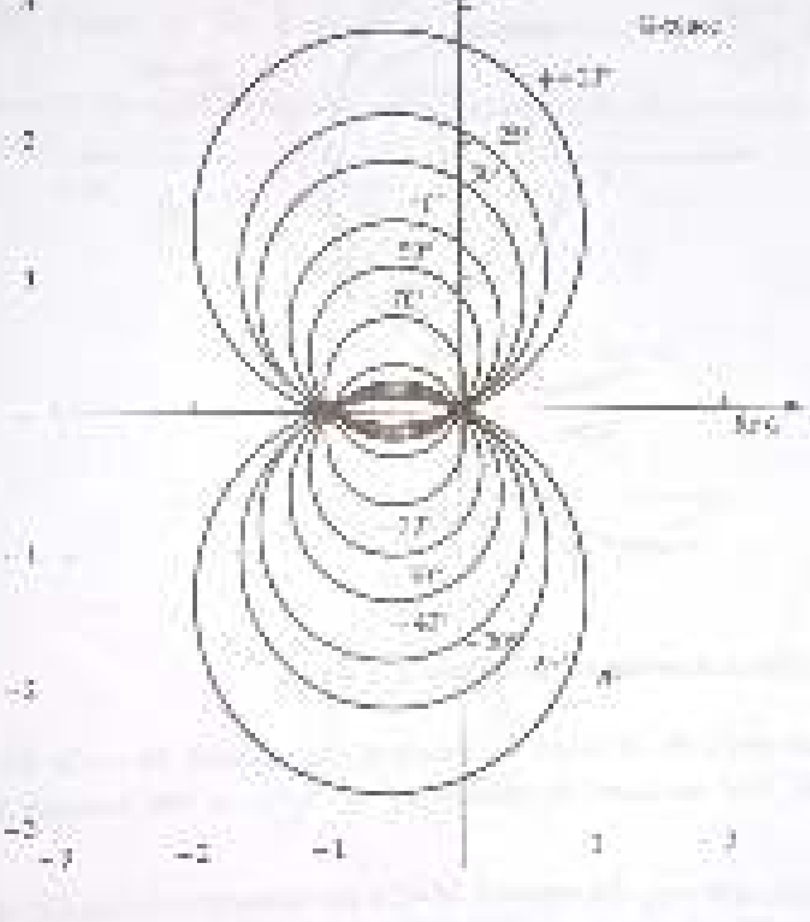


Fig. 11.20 A family of constant-N circles

Scanned by CamScanner