Continuity of a function

Definition

A function $f(x)$ is said to be continuous at $x=a$ if it satisfy the following conditions.

(i) $\lim_{x\to a} f(x)$ exists

(ii) $f(a)$ is defined.

(iii) $\lim_{x\to a} f(x) = f(a)$

If one or more of the above condition fail, the function $f(x)$ is said to be discontinuous at $x=a$.

Continuous function

A function is said to be continuous if it is continuous at each point of its domain.

Working procedure for testing continuity at a point $x=a$

1st step → First find $\lim_{x\to a} f(x)$ by using previous chapter.

If $\lim_{x\to a} f(x)$ does not exist then $f(x)$ is discontinuous at $x=a$.

If $\lim_{x\to a} f(x) = l$, then go to 2nd step.

2nd step → Find $f(a)$ from the given data.

If $f(a)$ is undefined then $f(x)$ is not continuous at $x=a$.

If $f(a)$ has definite value then go to 3rd step.

3rd step - Compare $\lim_{x \to a} f(x)$ and $f(a)$

If $\lim_{x \to a} f(x) = f(a)$, then $f(x)$ is continuous at $x = a$, otherwise $f(x)$ is discontinuous at $x = a$.

Examples

Q.1. Examine the continuity of the function $f(x)$ at $x = 3$.

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$$

Ans

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x \to 3} (x+3) \quad \left\{ \begin{array}{l} \text{As } x \to 3 \\ x \neq 3 \Rightarrow x-3 \neq 0 \end{array} \right.$$

$$= 3+3 = 6$$

From given data $f(3) = 6$

Now from above $\lim_{x \to 3} f(x) = f(3)$

$\therefore$ $f(x)$ is continuous at $x = 3$

Q.2. Test continuity of $f(x)$ at '0'.

$$f(x) = \begin{cases} (1+3x)^{\frac{1}{x}} & x \neq 0 \\ e^3 & x = 0 \end{cases}$$

Ans

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+3x)^{\frac{1}{x}} = \lim_{x \to 0} (1+3x)^{\frac{1}{3x} \cdot 3}$$

$$= \lim_{x \to 0} \left\{ (1+3x)^{\frac{1}{3x}} \right\}^3 = \left\{ \lim_{x \to 0} (1+3x)^{\frac{1}{3x}} \right\}^3 = e^3$$

As $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \Rightarrow \lim_{x\to 0}(1+\lambda x)^{\frac{1}{\lambda x}} = e$

In particular $\lim_{x\to 0}(1+3x)^{\frac{1}{3x}} = e$

And we know $\lim_{x\to a}\{f(x)\}^n = \{\lim_{x\to a} f(x)\}^n$

From given data $f(0) = e^3$

Hence $\lim_{x\to 0} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x=0$.

Q.3. Test continuity of $f(x)$ at $x=0$.

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x=0 \end{cases}$$

Ans

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{|x|}{x}$

As $|x|$ is present so we have to evalute the above limit by L.H.L and R.H.L. method.

$L.H.L = \lim_{x\to 0^-}\frac{|x|}{x}$ $\left\{\begin{array}{l} x\to 0^- \\ \Rightarrow x<0 \end{array}\right\}$

$= \lim_{x\to 0^-}\frac{-x}{x} = \lim_{x\to 0^-}(-1) = (-1)$

$R.H.L = \lim_{x\to 0^+}\frac{|x|}{x}$ $\left\{\begin{array}{l} x\to 0^+ \\ \Rightarrow x>0 \end{array}\right.$

$= \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+}(1) = 1$

Hence $L.H.L \neq R.H.L$

$\therefore$ $\lim_{x\to 0} f(x)$ does not exist

$\Rightarrow$ $f(x)$ is not continuous at $x=0$

4) Test continuity of $\frac{x^2-4}{x-2}$ at $x=2$

Ans

Here $f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$ undefined

Hence $f(x)$ is not continous at $x=2$

5) Test continuity of $f(x)$ at '0'.

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan 5x} & x \neq 0 \\ \frac{5}{3} & x = 0 \end{cases}$$

Ans

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 3x}{\tan 5x} = \lim_{x\to 0} \frac{\frac{\sin 3x}{x}}{\frac{\tan 5x}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 3x}{3x}\cdot 3}{\frac{\tan 5x}{5x}\cdot 5}$$

$$= \frac{3}{5} \lim_{x\to 0} \left\{ \left(\frac{\sin 3x}{3x}\right) \Big/ \left(\frac{\tan 5x}{5x}\right) \right\}$$

$$= \frac{3}{5}\left(\frac{1}{1}\right) = \frac{3}{5}$$

Given that $f(0) = \frac{5}{3}$

Thus $\lim_{x\to 0} f(x) \neq f(0)$

Hence $f(x)$ is not continuous at $x=0$.

Q6 Test continuity of $f(x)$ at $x=\frac{1}{2}$

$$f(x) = \begin{cases} 1-x & , x \leq \frac{1}{2} \\ x & , x > \frac{1}{2} \end{cases}$$

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Ans

First Understand the function properly

when $x < \frac{1}{2}$, $f(x) = 1-x$

$x > \frac{1}{2}$, $f(x) = x$

when $x = \frac{1}{2}$, $f(x) = 1-x = 1-\frac{1}{2} = \frac{1}{2}$

Now let us find the $\lim_{x \to \frac{1}{2}} f(x)$

$L.H.L = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} (1-x)$

$\left\{ \begin{array}{l} \text{As } x \to \frac{1}{2}^- \\ x < \frac{1}{2} \\ \text{So } f(x) = 1-x \\ \text{from above} \end{array} \right.$

$= 1 - \frac{1}{2} = \frac{1}{2}$

$R.H.L = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} x$

$\left\{ \begin{array}{l} \text{As } x \to \frac{1}{2}^+ \\ \Rightarrow x > \frac{1}{2} \\ \Rightarrow f(x) = x \end{array} \right.$

$= \lim_{x \to \frac{1}{2}^+} x = \frac{1}{2}$

Now from above as $L.H.L = R.H.L$

$\Rightarrow \boxed{\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}}$ — (1)

From defn $\boxed{f\left(\frac{1}{2}\right) = \frac{1}{2}}$ — (2)

From (1) and (2)

$$\lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$$

Hence $f(x)$ is continuous at $x = \frac{1}{2}$

Q.7. Test continuity of $f(x)$ at $x = 0, 1$.

At

$$f(x) = \begin{cases} 2x+1, & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$$

Ans

Here given that

$f(x) = 2x+1$ for $x < 0$ —— (1)

when $x = 0$, $f(x) = f(0) = 2x+1 = 2 \times 0 + 1 = 1$ —— (2)

when $0 < x < 1$, $f(x) = x$ —— (3)

when $x = 1$, $f(x) = f(1) = x = 1$ —— (4)

when $x > 1$, $f(x) = 2x - 1$ —— (5)

Continuity test at $x = 0$

$$\begin{aligned} L.H.L. &= \lim_{x \to 0^-} f(x) \\ &= \lim_{x \to 0^-} (2x+1) \\ &= 2 \times 0 + 1 = 1 \end{aligned} \quad \left\{ \begin{array}{l} x \to 0^- \\ \Rightarrow x < 0 \\ \Rightarrow f(x) = 2x+1 \\ \quad \text{from (1)} \end{array} \right.$$

$$\begin{aligned} R.H.L. &= \lim_{x \to 0^+} f(x) \\ &= \lim_{x \to 0^+} x = 0 \end{aligned} \quad \left\{ \begin{array}{l} x \to 0^+ \Rightarrow x > 0 \\ \Rightarrow 0 < x < 1 \\ \Rightarrow f(x) = x \end{array} \right.$$

As $L.H.L \neq R.H.L$.

$\Rightarrow \lim_{x \to 0} f(x)$ does not exist

Hence $f(x)$ is not continuous at $x = 0$

Continuity test at $x=1$

$L.H.L = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x$

$= \lim_{x\to 1^-} x = 1$

$\left\{ \begin{array}{l} x\to 1^- \\ \Rightarrow x<1 \\ \text{ie. } 0<x<1 \\ \Rightarrow f(x)=x \\ \quad \text{from(3)} \end{array} \right.$

$R.H.L. = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x-1$

$\left\{ \begin{array}{l} x\to 1^+ \Rightarrow x>1 \\ \Rightarrow f(x)=2x-1 \\ \quad \text{from (5)} \end{array} \right.$

$= 2\times 1 - 1 = 1$

As $L.H.L = R.H.L. \Rightarrow \boxed{\lim_{x\to 1} f(x) = 1}$

from given data $f(1) = 1$ (Eqⁿ(4))

Hence $\boxed{\lim_{x\to 1} f(x) = f(1)}$

$\therefore$ $f(x)$ is continuous at $x=1$

8. Examine continuity of $f(x)=[3x+11]$ at $x=-\frac{11}{3}$

Ans

$\lim_{x\to -\frac{11}{3}} f(x) = \lim_{x\to -\frac{11}{3}} [3x+11]$

$= \lim_{u\to 0} [u] \quad \text{——(1)}$

$\left\{ \begin{array}{l} \text{Let} \\ u=3x+11 \\ \text{when } x\to -\frac{11}{3} \\ u\to 3\times -\frac{11}{3}+11 \\ \text{ie. } u\to 0 \end{array} \right.$

Now $\lim_{u\to 0^-} [u] = \lim_{u\to 0^-} (-1)$

$= -1$

$\left\{ \begin{array}{l} \text{As } u\to 0^- \\ \Rightarrow -1<u<0 \\ \Rightarrow [u]=-1 \end{array} \right.$

$\lim_{u\to 0^+} [u] = \lim_{u\to 0^+} 0 = 0$

$\left\{ \begin{array}{l} \text{As } u\to 0^+ \\ \Rightarrow 0<u<1 \\ \Rightarrow [u]=0 \end{array} \right.$

As $L.H.L. \neq R.H.L.$

Hence $f(x)$ is not continuous at $x=0$

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9. Determine the value of K for which $f(x)$ is continuous at $x=1$

$$f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-1} & x \neq 1 \\ K. & x=1 \end{cases}$$

Ans Given function is continous at $x=1$

$\Rightarrow \lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \boxed{\lim_{x\to 1} f(x) = K}$ —— (1)

Now let us find $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-3x+2}{x-1} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1} \frac{x^2-2x-x+2}{x-1}$$

$$= \lim_{x\to 1} \frac{x(x-2)-1(x-2)}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-2)(x-1)}{(x-1)} \quad \left\{\begin{matrix} \text{As } x\to 1 \\ x\neq 1 \\ \Rightarrow x-1\neq 0 \end{matrix}\right\}$$

$$= \lim_{x\to 1} (x-2)$$

$$= 1-2 = -1 \quad \text{—— (2)}$$

from (1) and (2) $\boxed{K=-1}$ (Ans)

10. If $f(x) = \begin{cases} ax^2+b & \text{if } x<1 \\ 1 & \text{if } x=1 \\ 2ax-b & \text{if } x>1 \end{cases}$

is continuous at $x=1$, then find a and b.

Ans

Given that $f(x)$ is continuous at $x=1$

$\Rightarrow \lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \boxed{\lim_{x\to 1} f(x) = 1}$ — (1) $\left\{ \text{as } \underset{\text{given}}{f(1) = 1} \right\}$

From (1) as $\lim_{x\to 1} f(x)$ exists

$\Rightarrow \boxed{\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)}$ — (2)

From (1) and (2) we have,

$\lim_{x\to 1^-} f(x) = 1$

$\Rightarrow \lim_{x\to 1^-} (ax^2+b) = 1$

$\Rightarrow a\times 1^2 + b = 1$

$\Rightarrow \boxed{a+b=1}$ — (3)

$\left\{ \begin{array}{l} \text{As } x\to 1^- \\ \Rightarrow x<1 \\ \Rightarrow f(x) = ax^2+b \\ \text{from def}^n \\ \text{of } f(x) \end{array} \right.$

Again from (1) and (2)

$\lim_{x\to 1^+} f(x) = 1$

$\Rightarrow \lim_{x\to 1^+} (2ax - b) = 1$

$\Rightarrow 2\times a\times 1 - b = 1$

$\Rightarrow \boxed{2a - b = 1}$ — (4)

$\left\{ \begin{array}{l} x\to 1^+ \\ \Rightarrow x>1 \\ \Rightarrow f(x) = 2ax-b \end{array} \right.$

Eq^n (3) $a + \cancel{b} = 1$

Eq^n (4) $2a - \cancel{b} = 1$

$3a = 2 \Rightarrow \boxed{a = \frac{2}{3}}$

From (2) $a+b=1$

$\Rightarrow b = 1-a = 1-\frac{2}{3} = \frac{1}{3}$

Hence $a = \frac{2}{3}$ and $b = \frac{1}{3}$.

Q.11. Find the value of 'a' such that

$$f(x) = \begin{cases} \frac{\sin ax}{\sin x} & x \neq 0 \\ \frac{1}{a} & x = 0 \end{cases}$$

is continuous at $x=0$

Ans $f(x)$ is continuous at $x=0$

$\Rightarrow \lim_{x\to 0} f(x) = f(0)$

$\Rightarrow \lim_{x\to 0} \frac{\sin ax}{\sin x} = \frac{1}{a}$

$\Rightarrow \lim_{x\to 0} \frac{a\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin x}{x}\right)} = \frac{1}{a}$

$\Rightarrow a \lim_{x\to 0} \frac{\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin x}{x}\right)} = \frac{1}{a}$

$\Rightarrow a \frac{1}{1} = \frac{1}{a}$

$\Rightarrow a^2 = 1 \Rightarrow \boxed{a = \pm 1}$. (Ans)

Q. 12. Examine the continuity of the function $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

at $x=0$.

Ans

Let us Evalute $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

We know that $-1 \leq \sin\frac{1}{x} \leq 1$

$\Rightarrow (-1)x^2 \leq x^2 \sin\frac{1}{x} \leq x^2 \cdot 1$

$\Rightarrow -x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$

Now $\lim_{x \to 0} (-x^2) = -0^2 = 0$

$\lim_{x \to 0} x^2 = 0^2 = 0$

Hence by Sandwich theorem

$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

Given $f(0) = 0$

$\therefore$ Hence $\lim_{x \to 0} f(x) = f(0)$

$\therefore$ $f(x)$ is continous at $x=0$.

Q.13. Test continuity of $f(x)$ at $x=0$

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ans

Evaluation of $\lim_{x \to 0} f(x)$ is not possible directly.

$LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ $\left\{ \begin{array}{l} \text{when } x \to 0^- \\ \frac{1}{x} \to -\infty \\ e^{\frac{1}{x}} \to 0 \end{array} \right.$

$= \dfrac{0-1}{0+1} = -1$

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$$R.H.L = \lim_{x\to 0^+} f(x)$$

When $x \to 0^+$

$\Rightarrow \frac{1}{x} \to \infty$

$\Rightarrow e^{\frac{1}{x}} \to \infty$

$\Rightarrow \frac{1}{e^{1/x}} \to 0$

$$= \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$= \lim_{x\to 0^+} \frac{\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} - \frac{1}{e^{\frac{1}{x}}}}{\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} + \frac{1}{e^{\frac{1}{x}}}}$$

$$= \lim_{x\to 0^+} \left(\frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}} \right)$$

$$= \frac{1-0}{1+0} = 1$$

From above $L.H.L \neq R.H.L$.

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

$\therefore$ $f(x)$ is not continuous at $x=0$.

14. Discuss the continuity of the function

$$f(x) = \begin{cases} x - \frac{|x|}{x} & x \neq 0 \\ 2 & x = 0 \end{cases} \quad \text{at } x=0.$$

Ans:

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x - \frac{|x|}{x}$$

$x \to 0^-$

$\Rightarrow x < 0$

$\Rightarrow |x| = -x$

$$= \lim_{x\to 0^-} \left(x - \left(\frac{-x}{x} \right) \right)$$

$$= \lim_{x\to 0^-} \{x - (-1)\} = \lim_{x\to 0^-} (x+1)$$

$$= 0+1 = 1$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x - \frac{|x|}{x}$$

$x \to 0^+$
$\Rightarrow x > 0$
$\Rightarrow |x| = x$

$$= \lim_{x\to 0^+} \left(x - \frac{x}{x}\right)$$

$$= \lim_{x\to 0^+} (x-1) = 0-1 = -1$$

So $L.H.L \neq R.H.L$.

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

$\therefore f(x)$ is not continuous at $x=0$.

Assignment problems

1) Find the value of the constant K, so that the function given below is continuous at $x=0$.

$$f(x) = \begin{cases} \frac{1-\cos 2x}{2x^2} & x \neq 0 \\ K & x=0 \end{cases}$$

2) Test the continuity of $f(x)$ at $x=1$.

where $f(x) = \begin{cases} x^2+1 & x<1 \\ 2 & x=1 \\ 3x-1 & x>1 \end{cases}$

3) Show that the function $f(x)$ given by

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & x \neq 0 \\ 2 & x=0 \end{cases}$$

is continuous at $x=0$.

4) Test continuity of $f(x)$ at $x=1$.

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 7 & x=1 \end{cases}$$

5) Test Continuity of $f(x)$ at $x=0$

$$f(x) = \begin{cases} (1+2x)^{\frac{1}{x}} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$$

6) Test continuity of $f(x)$ at $x=2$

$$f(x) = \begin{cases} \dfrac{|x-2|}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

7) Find the value of K for which $f(x)$ is continuous at $x=0$

$$f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1}{x^2} & x \neq 0 \\ K & x = 0 \end{cases}$$

8) Test the continuity of the function $f(x)$ at $x=0$

$$f(x) = \begin{cases} \dfrac{\sin 3x}{\tan^{-1} 7x} & x \neq 0 \\ \dfrac{3}{7} & x = 0 \end{cases}$$

9) Test continuity of the function $f(x)$ at $x=1$, $f(x) = \begin{cases} \dfrac{x^2-4x+3}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

10) Examine the continuity of the function $f(x)$ at $x=1$.

$$f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ x^2-1 & \text{if } x > 1 \end{cases}$$